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Merging in maps and in pavings

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Abstract

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We introduce here a notion of paving extending the notion of map, which allows an easy definition of the incidence relations between vertices, edges, faces and pieces. More precisely, a paving is a quadruple $P=(B, \alpha, \sigma, \varphi)$ where B is a finite set and α, σ, φ are permutations of B such that $\alpha^2=1$, $\varphi\circ\alpha\circ\varphi=\alpha$, $\sigma\circ\varphi^{-1}\circ\sigma=\varphi$ and, $\forall x\in B$, $\alpha(x)\neq x$ and $\varphi(x)\notin\{\alpha(x), \sigma(x)\}$. If r, s, m and n are respectively the numbers of pieces, faces, edges and vertices of P, then $\chi(P)=r-s+m-n$ is called the characteristic of P. $C=(B, \alpha, \sigma)$ is a map and, if s', m', n' are respectively the numbers of faces, edges and vertices of C, then $g(C)=r+\frac{1}{2}(-s'+m'-n')$ is called the genus of C. The merging of two pieces according to a common face is a fundamental operation in pavings. The simultaneous studies of merging in maps and in pavings has given a result which links $\chi(P)$ and $g(C)$.

Introduction

Modeling a finite cell complex in the Euclidean space E^3 of dimension 3 or an assembling of a finite number of solids which are stuck face to face is fundamental in computational geometry and has applications in each domain dealing with representations in E^3.

Data structures for modeling a solid by a boundary representation have been proposed by Baumgart [3], Requicha [17] and by Ansaldi, De Floriani and Falcidieno [1] and results on Euler operations are given by Mäntylä and Sulonen [16]. Guibas and Stolfi [11] have defined an edge-algebra and a quad-edge data structure for representing subdivisions on surfaces. This has been generalized to subdivisions in the Euclidean space E^3 by Dobkin and Laszlo [6].

Other methods for modeling are based on the notion of map introduced by Edmonds (1960) and elaborated firstly by Cori [5]. Maps can be used to represent orientable surfaces. A notion of unoriented map which is able to represent orientable or not orientable surfaces has been given by Tutte [20]. Lienhardt [13, 14] has introduced a notion of V-map which generalizes the notion of map to the three-dimensional space E^3 and the notion of paving introduced by Arquès and Koch [2] is another generalization of the same notion which is directly associated to a subdivision of E^3.

Recently Lienhardt [15] and Brisson [4] have given extensions to the Euclidean space of dimension d and Dufourd [7] has given an algebraic specification of generalized maps.

The notion of paving introduced here is a generalization of maps other than those of Lienhardt and of Arquès and Koch. In our paving, every cell of an oriented subdivision of E^3 is represented by a map. Fontet [10] has introduced the operation of merging two disjoint maps. By extending this definition, we have proved some results on the genus of the transformed map. Similarly, merging a paving according to a face f is very simple if the two pieces p_1 and p_2 which have f in common are distinct but there are remarkable properties when $p_1 = p_2$. Such mergings are very natural and occur necessarily if we merge successively the pieces along all faces around an edge.

In the Sections 1-4 we give generalities on pavings. Section 5 deals with merging in maps and Section 6 with merging in pavings.

1. Maps and pavings

Definition 1.1. (i) Every $C = (B, \alpha, \sigma)$ where B is a finite set, α and involution of B without fixed points ($\forall b \in B$, $\alpha(b) \neq b$ and $\alpha^2(b) = b$) and σ a permutation of B, is called a *map*. The elements of B are called *darts*.

(ii) For every set Θ of permutations of B, let $\langle\Theta\rangle$ be the group of permutations of B generated by Θ, $\forall b \in B$, let $\langle\Theta\rangle(b) = \{\vartheta(b);\ \vartheta \in \langle\Theta\rangle\}$ be the orbit of b relative to the group $\langle\Theta\rangle$ and $\zeta(\Theta)$ the number of orbits of $\langle\Theta\rangle$ in B.

(iii) $\forall b \in B$, the *darts* b and $\alpha(b)$ are said to be *opposite*;

- $\langle\alpha\rangle(b) = \{b, \alpha(b)\}$ is called the *edge* of b in C;
- $\langle\sigma\rangle(b) = \{b, \sigma(b), \ldots, \sigma^{k-1}(b)\}$ where k is the smallest integer such that $\sigma^k(b) = b$ is called the *vertex* of b;
- $\mathrm{df}(b) = \langle\sigma^{-1} \circ \alpha\rangle(b)$ is called the *direct face* of b and $\langle\sigma \circ \alpha\rangle(b)$ is called the *retrograd face* of b.

(iv) $g(C) = \zeta(\alpha, \sigma) + \frac{1}{2}[-\zeta(\sigma^{-1} \circ \alpha) + \zeta(\alpha) - \zeta(\sigma)]$ where $\zeta(\alpha, \sigma)$ is the number of connected components of C and $\zeta(\sigma^{-1} \circ \alpha)$, $\zeta(\alpha)$ and $\zeta(\sigma)$ are respectively the numbers of faces, edges and vertices of C, is called the *genus* of C: $g(C) \geqslant 0$ and $g(C)$ is an integer (see [12]). A *map* of genus 0 is said to be *planar*.

(v) The multigraph $G(C)$ where the vertices and the edges are those of map C is called the *graph of map C*.

Example 1.2. If $C = (B, \alpha, \sigma)$ where $\sigma = (1, -4, -5)(2, -1)(3, -2, -6)(4, -3)(5, 6)$ and $\forall x \in B$, $\alpha(x) = -x$, then $\langle\sigma\rangle(1) = \{1, -4, -5\}$ is a vertex, $\mathrm{df}(1) = \{1, 2, 3, 4\}$ is a direct face and $g(C) = 0$ (see Fig. 1).

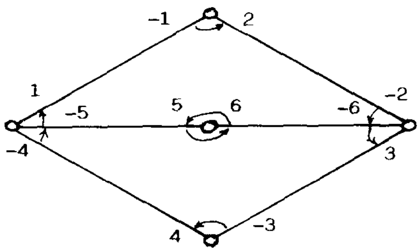


Fig. 1.

Definition 1.3. (i) Every $P=(B, \alpha, \sigma, \varphi)$, where $C=(B, \alpha, \sigma)$ is a map and φ is a permutation of B such that

(1) $\varphi \circ \alpha \circ \varphi = \alpha$,

(2) $\sigma \circ \varphi^{-1} \circ \sigma = \varphi$,

(3) $\forall x \in B,\ \varphi(x) \notin \{\alpha(x), \sigma(x)\}$,

is called a *paving*.

(1) is equivalent to $\alpha \circ \varphi = \varphi^{-1} \circ \alpha$ and to $\varphi \circ \alpha = \alpha \circ \varphi^{-1}$ and, since $\alpha^{-1} = \alpha$, to $\alpha \circ \varphi$ (resp. $\varphi \circ \alpha$) being an involution.

(2) is equivalent to $\sigma \circ \varphi^{-1} = \varphi \circ \sigma^{-1}$, to $\sigma^{-1} \circ \varphi = \varphi^{-1} \circ \sigma$, to $\varphi \circ \sigma^{-1} \circ \varphi = \sigma$ and to $\varphi \circ \sigma^{-1}$ (resp. $\varphi^{-1} \circ \sigma$) being an involution.

(3) signifies that none of the involutions $\alpha \circ \varphi$, $\varphi \circ \alpha$, $\varphi \circ \sigma^{-1}$ and $\varphi^{-1} \circ \sigma$ has a fixed point.

(ii) The *map* $C=(B, \alpha, \sigma)$ is said to be *underlying to paving* $P=(B, \alpha, \sigma, \varphi)$.

Lemma 1.4. *For every dart b of a paving $P=(B, \alpha, \sigma, \varphi)$,*

(i) $\alpha(\langle\varphi\rangle(b)) = \langle\varphi\rangle(\alpha(b))$ *and* $\langle\alpha, \varphi\rangle(b) = \langle\varphi\rangle(b) \cup \langle\varphi\rangle(\alpha(b))$;

(ii) $\varphi^{-1} \circ \sigma(\langle\sigma^{-1} \circ \alpha\rangle(b)) = \langle\sigma^{-1} \circ \alpha\rangle(\varphi^{-1} \circ \sigma(b))$ *and*

$$\langle\varphi^{-1} \circ \sigma, \sigma^{-1} \circ \alpha\rangle(b) = \langle\sigma^{-1} \circ \alpha\rangle(b) \cup \langle\sigma^{-1} \circ \alpha\rangle(\varphi^{-1} \circ \sigma(b)).$$

(iii) *If* $f = \langle\varphi^{-1} \circ \sigma, \sigma^{-1} \circ \alpha\rangle(b)$ *then* $\varphi(f) = \sigma(f) = \alpha(f)$.

Proof. (i) $\forall x \in \langle\varphi\rangle(b)$, $\exists i \in \mathbb{N}$ such that $x = \varphi^i(b)$ and, since $\alpha \circ \varphi = \varphi^{-1} \circ \alpha$ implies

$$\alpha \circ \varphi^i = \varphi^{-i} \circ \alpha,\ \alpha(x) = \alpha \circ \varphi^i(b) = \varphi^{-i} \circ \alpha(b) \in \langle\varphi\rangle(\alpha(b)).$$

Thus $\alpha(\langle\varphi\rangle(b)) \subset \langle\varphi\rangle(\alpha(b))$. Similarly, $\alpha(\langle\varphi\rangle(\alpha(b)) \subset \langle\varphi\rangle(\alpha^2(b)) = \langle\varphi\rangle(b)$ and hence $\alpha(\langle\varphi\rangle(b)) = \langle\varphi\rangle(\alpha(b))$. It follows that $\langle\varphi\rangle(b) \cup \langle\varphi\rangle(\alpha(b))$ is stable for $\langle\alpha, \varphi\rangle$ and hence that $\langle\alpha, \varphi\rangle(b) = \langle\varphi\rangle(b) \cup \langle\varphi\rangle(\alpha(b))$.

(ii) $\forall x \in \langle\sigma^{-1} \circ \alpha\rangle(b)$, $\exists i \in \mathbb{N}$ such that $x = (\sigma^{-1} \circ \alpha)^i(b)$ and hence

$$\begin{aligned}(\varphi^{-1} \circ \sigma) \circ (\sigma^{-1} \circ \alpha) &= \varphi^{-1} \circ \alpha = \alpha \circ \varphi = \alpha \circ \sigma \circ \sigma^{-1} \circ \varphi \\ &= (\sigma^{-1} \circ \alpha)^{-1} \circ (\varphi^{-1} \circ \sigma),\end{aligned}$$

$$\begin{aligned}\varphi^{-1} \circ \sigma(x) &= \varphi^{-1} \circ \sigma \circ (\sigma^{-1} \circ \alpha)^i(b) \\ &= (\sigma^{-1} \circ \alpha)^{-i} \circ (\varphi^{-1} \circ \sigma)(b) \in \langle\sigma^{-1} \circ \alpha\rangle(\varphi^{-1} \circ \sigma)(b).\end{aligned}$$

Thus $\varphi^{-1}\circ\sigma(\langle\sigma^{-1}\circ\alpha\rangle(b))\subset\langle\sigma^{-1}\circ\alpha\rangle(\varphi^{-1}\circ\sigma(b))$. Since $\varphi^{-1}\circ\sigma$ is an involution, by symmetry, we have

$$\varphi^{-1}\circ\sigma(\langle\sigma^{-1}\circ\alpha\rangle(b))=\langle\sigma^{-1}\circ\alpha\rangle(\varphi^{-1}\circ\sigma(b)).$$

Thus $\langle\sigma^{-1}\circ\alpha\rangle(b)\cup\langle\sigma^{-1}\circ\alpha\rangle(\varphi^{-1}\circ\sigma(b))$ is stable for $\langle\sigma^{-1}\circ\alpha,\varphi^{-1}\circ\sigma\rangle$ and hence

$$\langle\varphi^{-1}\circ\sigma,\sigma^{-1}\circ\alpha\rangle(b)=\langle\sigma^{-1}\circ\alpha\rangle(b)\cup\langle\sigma^{-1}\circ\alpha\rangle(\varphi^{-1}\circ\sigma(b)).$$

(iii) By (ii), $\varphi^{-1}\circ\sigma(f)=\sigma^{-1}\circ\alpha(f)=f$ and hence $\varphi(f)=\sigma(f)=\alpha(f)$. □

Definition 1.5. (i) $\forall b\in B$, the connected component $p(b)=\langle\sigma,\alpha\rangle(b)$ of map $C=(B,\alpha,\sigma)$ is called the *piece* of b. Thus C is the disjoint union of the pieces of P.

- $s(b)=\langle\sigma,\varphi\rangle(b)$ is called the *vertex* of b;
- $\langle\varphi\rangle(b)$ is called the *sheaf* of b;
- $a(b)=\langle\alpha,\varphi\rangle(b)=\langle\varphi\rangle(b)\cup\langle\varphi\rangle(\alpha(b))$ is called the *edge* of b;
- $\mathrm{df}(b)=\langle\sigma^{-1}\circ\alpha\rangle(b)$ is called the *half-face* of b: $\mathrm{df}(b)$ is the direct face of $p(b)$;
- $f(b)=\langle\varphi^{-1}\circ\sigma,\sigma^{-1}\circ\alpha\rangle(b)=\mathrm{df}(b)\cup\mathrm{df}(\varphi^{-1}\circ\sigma(b))$ is called the *face* of b: the *half-face* $\mathrm{df}(b)$ of $p(b)$ is said to be *stuck on the half-face* $\mathrm{df}(\varphi^{-1}\circ\sigma(b))$ of $p(\varphi(b))$, and face $f(b)$ is said to be *common to the pieces* $p(b)$ and $p(\varphi(b))$.

(ii) Two elements of the set of all pieces, faces, edges and vertices are said to be *incident* if they have at least one common dart.

Definition 1.6. (i) The multigraph $G(P)$, where the vertices and the edges are those of paving P, is called the *graph of paving P*.

(ii) Paving P is said to be *connected* if graph $G(P)$ is connected or equivalently if the group $\langle\alpha,\sigma,\varphi\rangle$ acts transitively on B.

Definition 1.7. (i) For every paving $P=(B,\alpha,\sigma,\varphi)$,

$$\chi(P)=\zeta(\alpha,\sigma)-\zeta(\varphi^{-1}\circ\sigma,\sigma^{-1}\circ\alpha)+\zeta(\alpha,\varphi)-\zeta(\sigma,\varphi),$$

where $\zeta(\alpha,\sigma)$, $\zeta(\varphi^{-1}\circ\sigma,\sigma^{-1}\circ\alpha)$, $\zeta(\alpha,\varphi)$ and $\zeta(\sigma,\varphi)$ are respectively the numbers of pieces, faces, edges and vertices of P, is called the *characteristic* of paving P. If P is the union of disjoint pavings $P_1,\ldots,P_k$, then $\chi(P)=\chi(P_1)+\cdots+\chi(P_k)$. This is a generalization of the notion of Euler's characteristic (see [9] and [18]).

2. Representations of pavings

Definition 2.1. (i) Let E be the three-dimensional Euclidean space with an infinity point ∞, $L(E)$ the set of all lines of E which are homeomorphic to the open interval $]0,1[$ of $\mathbb{R}$, $S(E)$ the set of all surfaces of E which are homeomorphic to the open unit disk of the plane and $V(E)$ the set of all volumes of E which are homeomorphic to the open unit ball of E.

For every subset X of E, we denote the boundary of X by $\delta(X)$.

A partition Π of E defined by a set T of points of E, a set L of lines of $L(E)$ such that $\delta(L)=T$, a set S of surfaces of $S(E)$ such that $\delta(S)=T\cup L$ and a set V

of volumes of $V(E)$ such that $\delta(V) = T \cup L \cup S$ is called a *subdivision* of E. Such a subdivision $\Pi = (T, L, S, V)$ is said to be *oriented* if $\forall v \in V$, $\delta(v)$ is an oriented surface and if $\forall u, v \in V$, the orientations of $\delta(u)$ and $\delta(v)$ are opposite on $\delta(u) \cap \delta(v)$.

(ii) Let $P = (B, \alpha, \sigma, \varphi)$ be a connected paving and

- π a mapping from B into E such that

$$\forall b \in B, \quad \pi^{-1}(\pi(b)) = s(b);$$

- λ a mapping from B into $L(E)$ such that

$$\forall b \in B, \quad \delta(\lambda(b)) = \pi(a(b)) \quad \text{and} \quad \lambda^{-1}(\lambda(b)) = a(b);$$

- μ a mapping from B into $S(E)$ such that

$$\forall b \in B, \quad \delta(\mu(b)) = \{\lambda(c);\ c \in \mathrm{df}(b)\} \cup \{\pi(c);\ c \in \mathrm{df}(b)\} \quad \text{and}$$
$$\mu^{-1}(\mu(b)) = f(b);$$

- ν a mapping from B into $V(E)$ such that

$$\forall b \in B,\ \delta(\nu(b)) = \{\mu(c);\ c \in p(b)\} \cup \{\lambda(c);\ c \in p(b)\}$$
$$\cup \{\pi(c);\ c \in p(b)\} \quad \text{and}$$
$$\nu^{-1}(\nu(b)) = p(b).$$

If $(\pi(B), \lambda(B), \mu(B), \nu(B))$ is a subdivision of E, then (π, λ, μ, ν) is called a *representation of paving P* in E. Such a subdivision is oriented since $\forall b \in B$, $p(b)$ is a map and a map can only represent orientable surfaces. This representation is able to separate the geometrical notions from the topological ones.

Remark. Arquès and Koch [2] have given some results on the problem of representability for another notion of paving but their results can be transposed to our pavings. Lienhardt's V-maps [15] can represent oriented and nonoriented subdivisions of E.

Definition 2.2. We also use a representation of P in which each line $\lambda(b)$ is replaced by a sheaf of $r = |\langle\varphi\rangle(b)|$ "parallel" lines respectively associated to the darts b, $\varphi(b), \ldots, \varphi^{r-1}(b)$ and their opposites. A dart b is then represented by a vector or an oriented arc $\vec{u}$ and $\alpha(b)$ is represented by $-\vec{u}$.

Moreover we assume the following:

(i) $\forall b \in B$, φ permutes the darts of $\langle\varphi\rangle(b)$ and the pieces $\{p(c);\ c \in \langle\varphi\rangle(b)\}$ counterclockwise around edge $a(b)$ oriented by b from $s(b)$ to $s(\alpha(b))$. An orthogonal section of such a sheaf is drawn in Fig. 2.

(ii) Equality $\varphi \circ \alpha \circ \varphi = \alpha$ is true if and only if the diagram of Fig. 3 is commutative.

(iii) For every vertex s of a piece p, we pick a unitary vector $\vec{u}(s)$ placed at point $\pi(s)$ and oriented from inside to outside of the solid $\nu(p)$ which represents p and we assume that σ orders the darts of s counterclockwise around $\vec{u}(s)$ (see Fig. 4).

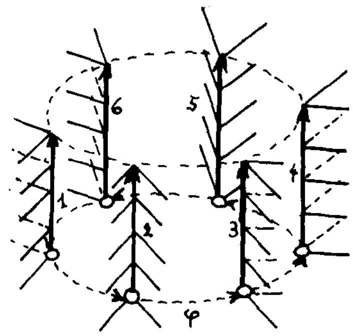


Fig. 2.

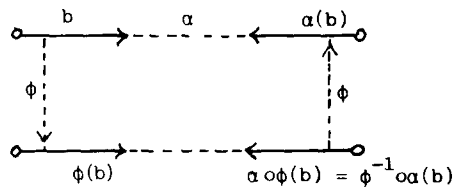


Fig. 3.

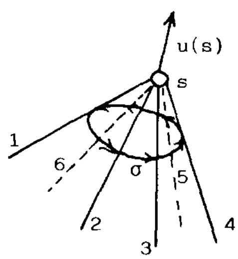


Fig. 4.

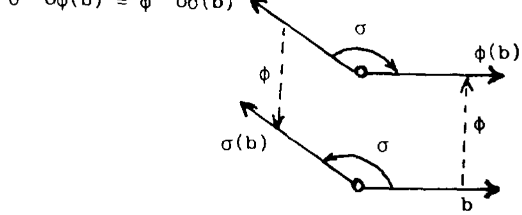


Fig. 5.

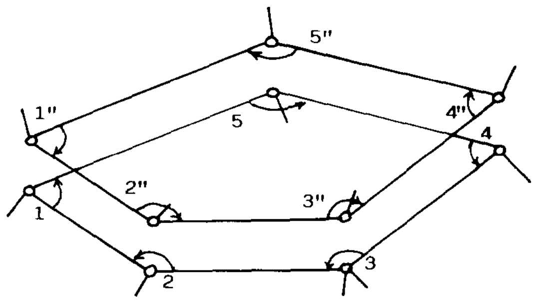


Fig. 6.

(iv) Equality $\varphi \circ \sigma^{-1} \circ \varphi = \sigma$ is then true if and only if the diagram of Fig. 5 is commutative. The half-faces of a face then have opposite orientations (see Fig. 6).

3. Examples of pavings

Example 3.1. Let P be the paving which has the pieces $p_1 = (B_1, \alpha_1, \sigma_1)$ with $B_1 = X_1 \cup -X_1$ where $X_1 = \{1, 2, 3, 4, 5, 6\}$, $\sigma_1 = (1, 2, 3)(-1, 4, -6)(-2, 6, -5)(-3, 5, -4)$ and $\alpha_1(x) = -x\ \forall x \in B_1$, (see Fig. 7a), $p_2 = (B_2, \alpha_2, \sigma_2)$ with $B_2 = X_2 \cup -X_2$

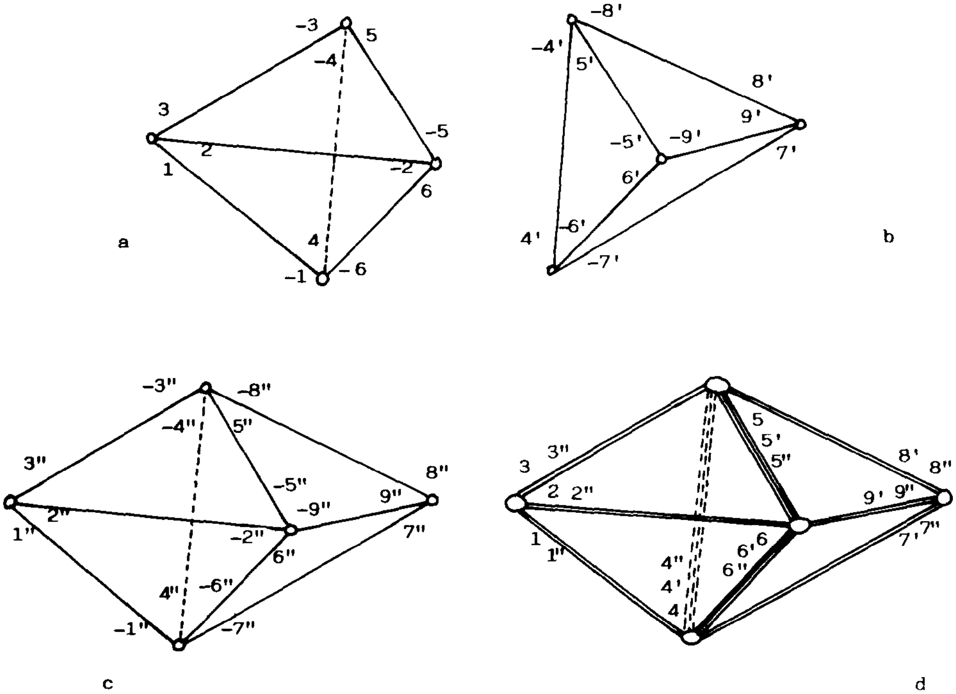


Fig. 7.

where $X_2 = \{4', 5', 6', 7', 8', 9'\}$,

$$\sigma_2 = (7', 8', 9')(-7', -6', 4')(-8', -4', 5')(-9', -5', 6')$$

and $\alpha_2(x) = -x\ \forall x \in B_2$ (see Fig. 7b) and $p_3 = (B_3, \alpha_3, \sigma_3)$ with $B_3 = X_3 \cup -X_3$ where $X_3 = \{1'', 2'', 3'', 4'', 5'', 6'', 7'', 8'', 9''\}$,

$$\sigma_3 = (1'', 3'', 2'')(-1'', -6'', -7'', 4'')(-2'', -5'', -9'', 6'') \circ (-3'', -4'', -8'', 5'')(7'', 9'', 8'')$$

and $\alpha_3(x) = -x\ \forall x \in B_3$ (see Fig. 7c),

$$\alpha = \alpha_1 \circ \alpha_2 \circ \alpha_3 \quad \text{and} \quad \sigma = \sigma_1 \circ \sigma_2 \circ \sigma_3 \quad \text{and} \quad \varphi = \varphi_1 \circ \varphi_2 \circ \varphi_3$$

where

$$\varphi_1 = (4, 4', 4'')(-4, -4'', -4')(5, 5', 5'')(-5, -5'', -5')(6, 6', 6'')(-6, -6'', -6'),$$

$\varphi_2 = \pi\{(x, x''); x \in \{1, 2, 3, -1, -2, -3\}$ and $\varphi_3 = \pi\{(x', x''); x \in \{7, 8, 9, -7, -8, -9\}\}$.

P then has 3 pieces, 7 faces, 9 edges and 5 vertices and thus $\chi(P) = 0$ (see Fig. 7d).

Definition 3.2. (i) Two maps $C_1 = (B_1, \alpha_1, \sigma_1)$ and $C_2 = (B_2, \alpha_2, \sigma_2)$ are said to be *isomorphic* if there exists a bijection ψ from B_1 onto B_2 such that $\alpha_2 \circ \psi = \psi \circ \alpha_1$ and $\sigma_2 \circ \psi = \psi \circ \sigma_1$.

(ii) Two maps $C_1 = (B_1, \alpha_1, \sigma_1)$ and $C_2 = (B_2, \alpha_2, \sigma_2)$ are said to be *inverse* if C_1 is isomorphic to the map $C_2^{-1} = (B_2, \alpha_2, \sigma_2^{-1})$, i.e. if there exists a one-to-one mapping ψ from B_1 onto B_2 such that $\alpha_2 \circ \psi = \psi \circ \alpha_1$ and $\sigma_2^{-1} \circ \psi = \psi \circ \sigma_1$.

(iii) If $C_1 = (B_1, \alpha_1, \sigma_1)$ and $C_2 = (B_2, \alpha_2, \sigma_2)$ are two inverse maps with $B_1 \cap B_2 = \emptyset$ and if ψ is an isomorphism from C_1 onto C_2^{-1}, let $B = B_1 \cup B_2$ and the permutations α, σ and φ of B be such that $\alpha|B_1 = \alpha_1$, $\alpha|B_2 = \alpha_2$, $\sigma|B_1 = \sigma_1$, $\sigma|B_2 = \sigma_2$, $\varphi|B_1 = \psi$ and $\varphi|B_2 = \psi^{-1}$. We then have $\varphi \circ \alpha \circ \varphi = \alpha$ and $\varphi \circ \sigma^{-1} \circ \varphi = \sigma$. Hence $P = (B, \alpha, \sigma, \varphi)$ is a paving. P is called the *canonical paving associated to* C_1 or to C_2.

Proposition 3.3. *If P is the canonical paving associated to a map C_1, then $\chi(P) = 2g(C_1)$.*

Proof. By Definition 1.7, $\chi(P) = \zeta(\alpha, \sigma) - \zeta(\varphi^{-1} \circ \sigma, \sigma^{-1} \circ \alpha) + \zeta(\alpha, \varphi) - \zeta(\sigma, \varphi)$. The number of pieces is equal to the number of connected components of map $C_1 \cup C_2$. Since C_1 and C_2 have the same number of connected components,

$$\zeta(\alpha, \sigma) = \zeta(\alpha_1, \sigma_1) + \zeta(\alpha_2, \sigma_2) = 2\zeta(\alpha_1, \sigma_1).$$

For every face $f = f(b)$ of P, $f \cap C_1 = \mathrm{df}(b)$ and $\psi(\mathrm{df}(b))$ is a retrograd face of C_2 and $\mathrm{df}(b) \to \psi(\mathrm{df}(b))$ is a bijection from the set of direct faces of C_1 onto the set of retrograd faces of C_2. Thus

$$\zeta(\varphi^{-1} \circ \sigma, \sigma^{-1} \circ \alpha) = \zeta(\sigma_1^{-1} \circ \alpha_1).$$

$$\forall b \in B_1, \qquad s(b) \cap B_1 = \langle\sigma\rangle(b), \quad s(b) \cap B_2 = \langle\sigma\rangle(\psi(b)),$$
$$a(b) \cap B_1 = \langle\alpha\rangle(b) \quad \text{and} \quad a(b) \cap B_2 = \langle\alpha\rangle(\psi(b)).$$

Thus $\zeta(\sigma, \varphi) = \zeta(\sigma_1)$ and $\zeta(\alpha, \varphi) = \zeta(\alpha_1)$. It follows that $\chi(P) = 2g(C_1)$. □

Example 3.4. If $C_1 = (B_1, \alpha_1, \sigma_1)$ is the connected map such that $B_1 = X \cup -X$ with $X = \{1, 2, 3, 4, 5\}$,

$$\sigma_1 = (1, -3, -5)(2, -1)(3, -2, 4)(5, -4)$$

and $\alpha_1(x) = -x$, $\forall x \in B_1$, the canonical paving associated to C_1 has 2 pieces, 3 faces $f(1)$, $f(4)$ and $f(-1)$, 5 edges and 4 vertices. Thus $\chi(P) = 0$ (see Fig. 17a).

Example 3.5. If we stick 13 cubic pieces such that their join consists of a layer with three columns and two holes (see Fig. 8) and if we add an external piece which wraps this join (this external piece is not drawn), we obtain a paving which has 14 pieces, 64 faces, 100 edges and 48 vertices and thus $\chi(P) = 2$.

More generally, if a layer of cubic pieces with three columns has k holes and is wrapped in a unique external piece, we obtain a paving P which has $5k+4$ pieces, $24k+16$ faces, $36k+28$ edges and $16k+16$ vertices. Thus $\chi(P) = k$.

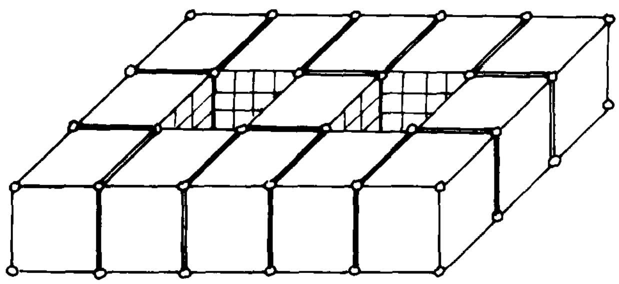


Fig. 8.

4. The dual of a paving

Proposition 4.1. *For every paving* $P = (B, \alpha, \sigma, \varphi)$, $P^{\perp} = (B, \varphi^{-1} \circ \sigma, \sigma^{-1}, \sigma^{-1} \circ \alpha)$ *is a paving whose pieces, faces, edges and vertices are respectively the vertices, edges, faces and pieces of P.*

Proof. (i) Let $\alpha' = \varphi^{-1} \circ \sigma$, $\sigma' = \sigma^{-1}$ and $\varphi' = \sigma^{-1} \circ \alpha$. By Definition 1.3, $\alpha' = \varphi^{-1} \circ \sigma$ is an involution and, since $\forall x \in B$, $\varphi(x) \neq \sigma(x)$, $\alpha'(x) \neq x$;

$$\begin{aligned}\varphi' \circ \alpha' \circ \varphi' &= \sigma^{-1} \circ \alpha \circ \varphi^{-1} \circ \sigma \circ \sigma^{-1} \circ \alpha \\ &= \sigma^{-1} \circ \alpha \circ \varphi^{-1} \circ \alpha = \sigma^{-1} \circ \varphi = \alpha'\end{aligned}$$

and

$$\sigma' \circ \varphi'^{-1} \circ \sigma' = \sigma^{-1} \circ \alpha^{-1} \circ \sigma \circ \sigma^{-1} = \sigma^{-1} \circ \alpha = \varphi'.$$

Since $\alpha(x) \neq \varphi(x)$,

$$\varphi'(x) = \sigma^{-1} \circ \alpha(x) \neq \sigma^{-1} \circ \varphi(x) = \alpha'(x)$$

and, since $\alpha(x) \neq x$,

$$\varphi'(x) = \sigma^{-1} \circ \alpha(x) \neq \sigma^{-1}(x) = \sigma'(x)$$

and hence $\varphi'(x) \notin \{\alpha'(x), \sigma'(x)\}$. Thus $P^{\perp}$ is a paving.

(ii) $\forall b \in B$, let $s^{\perp}(b)$ be the vertex, $a^{\perp}(b)$ the edge, $f^{\perp}(b)$ the face and $p^{\perp}(b)$ the piece of b in $P^{\perp}$.

- The vertices of $P^{\perp}$ are the pieces of P since $\langle \sigma', \varphi' \rangle = \langle \sigma^{-1}, \sigma^{-1} \circ \alpha \rangle = \langle \alpha, \sigma^{-1} \rangle = \langle \alpha, \sigma \rangle$ and thus $s^{\perp}(b) = p(b)$.
- The edges of $P^{\perp}$ are the faces of P since $\langle \alpha', \varphi' \rangle = \langle \varphi^{-1} \circ \sigma, \sigma^{-1} \circ \alpha \rangle$ and thus $a^{\perp}(b) = f(b)$.
- The faces of $P^{\perp}$ are the edges of P since $\langle \varphi'^{-1} \circ \sigma', \sigma'^{-1} \circ \alpha' \rangle = \langle \alpha \circ \sigma \circ \sigma^{-1}, \sigma \circ \varphi^{-1} \circ \sigma \rangle = \langle \alpha, \varphi \rangle$ and hence $f^{\perp}(b) = a(b)$.
- The pieces of $P^{\perp}$ are the vertices of P since $\langle \alpha', \sigma' \rangle = \langle \varphi^{-1} \circ \sigma, \sigma^{-1} \rangle = \langle \sigma^{-1}, \varphi^{-1} \rangle = \langle \sigma, \varphi \rangle$ and hence $p^{\perp}(b) = s(b)$. $\square$

Definition 4.2. $P^{\perp} = (B, \alpha', \sigma', \varphi')$ is called *the dual paving* of P.

Remark. The notion of duality has also been introduced by Arquès and Koch [2], Lienhardt [15], and Dobkin and Laszlo [6].

Example 4.3 (see Fig. 9). The dual paving $P^{\perp} = (B, \alpha', \sigma', \varphi')$ of paving P of Example 3.1 is such that

$$\begin{aligned} \alpha' = \varphi^{-1} \circ \sigma = {} & (1, 2'')(-1, 4'')(2, 3'')(-2, 6'')(3, 1'')(-3, 5'')(4, -6')(-4, -3'') \\ & \circ (5, -4')(-5, -2'')(6, -5')(-6, -1'')(4', -7'')(5', -8'') \\ & \circ (6', -9'')(7', 8'')(-7', -6'')(8', 9'')(-8', -4'') \\ & \circ (9', 7'')(-9', -5''), \end{aligned}$$

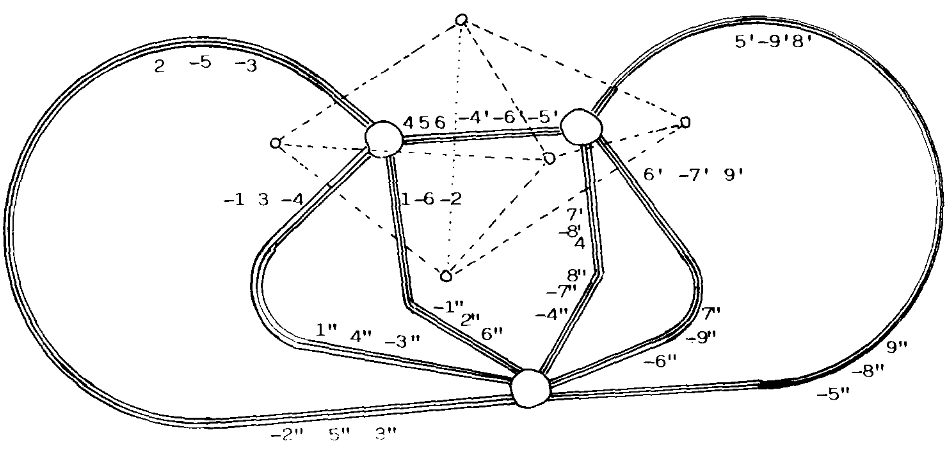


Fig. 9.

$$\sigma' = \sigma^{-1} = (1, 3, 2)(-1, -6, 4)(-2, -5, 6)(-3, -4, 5)(7', 9', 8')(-7', 4', -6') \circ (-9', 6', -5')(-8', 5', -4')(1'', 2'', 3'')(-1'', 4'', -7'', -6'') \circ (-2'', 6'', -9'', -5'')(-3'', 5'', -8'', -4'')(7'', 8'', 9'')$$

and

$$\varphi' = \sigma^{-1} \circ \alpha = (1, -6, -2)(-1, 3, -4)(2, -5, -3)(4, 5, 6)(4', -8', 7') \circ (5', -9', 8')(6', -7', 9')(-4', -6', -5')(1'', 4'', -3'')(-1'', 2'', 6'') \circ (-2'', 3'', 5'')(-4'', -7'', 8'')(-5'', -8'', 9'')(-6'', -9'', 7'').$$

$P^\perp$ has 5 pieces:

$$p^\perp(1) = \{1, 2, 3, 1'', 2'', 3''\} = s(1),$$
$$p^\perp(4) = \{4, -1, -6, 4', -6', -7', 4'', -1'', -6'', -7''\} = s(4),$$
$$p^\perp(5) = \{5, -3, -4, 5', -4', -8', 5'', -3'', -4'', -8''\} = s(5),$$
$$p^\perp(6) = \{6, -2, -5, 6', -5', -9', 6'', -2'', -5'', -9''\} = s(6)$$

and

$$p^\perp(7') = \{7', 8', 9', 7'', 8'', 9''\} = s(7').$$

Proposition 4.4. *For every paving P,* (i) $(P^\perp)^\perp = P$; (ii) $P^\perp$ *is connected if and only if P is connected*; (iii) $\chi(P) + \chi(P^\perp) = 0$.

Proof. (i) If $\alpha' = \varphi^{-1} \circ \sigma$, $\sigma' = \sigma^{-1}$ and $\varphi' = \sigma^{-1} \circ \alpha$, then

$$\alpha'' = \varphi'^{-1} \circ \sigma' = \alpha \circ \sigma \circ \sigma^{-1} = \alpha, \qquad \sigma'' = \sigma'^{-1} = \sigma$$

and

$$\varphi'' = \sigma'^{-1} \circ \alpha' = \sigma \circ \varphi^{-1} \circ \sigma = \sigma \circ \sigma^{-1} \circ \varphi = \varphi.$$

Thus $(P^\perp)^\perp = P$.

(ii) $P^\perp$ is connected if and only if the group $\langle \alpha', \sigma', \varphi' \rangle$ is transitive on B, that is if and only if P is connected since

$$\langle \alpha', \sigma', \varphi' \rangle = \langle \varphi^{-1} \circ \sigma, \sigma^{-1}, \sigma^{-1} \circ \alpha \rangle = \langle \varphi^{-1}, \sigma^{-1}, \alpha \rangle = \langle \alpha, \sigma, \varphi \rangle$$

(iii) This result follows immediately by Definition 1.7 and Proposition 4.1. □

Remark 4.5. By Example 3.5, $\chi(P)$ can be any integer and, following Proposition 4.4, $\chi(P)$ can be any rational integer.

5. Merging in maps

Lemma 5.1. *If b and c are darts of a map* $C = (B, \alpha, \sigma)$ *such that* $|\mathrm{df}(b)| = |\mathrm{df}(c)|$, *there exists a one-to-one mapping* ψ *of* $f = \mathrm{df}(b) \cup \mathrm{df}(c)$ *onto* $\alpha(f)$ *such that* $\psi(b) = \sigma(c)$, $\psi(c) = \sigma(b)$ *and,*

$$\forall x \in f, \qquad \psi^{-1} \circ \sigma(x) = \sigma^{-1} \circ \psi(x) \quad \textit{and} \quad \psi^{-1} \circ \alpha(x) = \alpha \circ \psi(x).$$

Moreover ψ *is unique.*

Proof. Let $\mathrm{df}(b)=\{b_0,\ldots,b_{k-1}\}$ and $\mathrm{df}(c)=\{c_0,\ldots,c_{k-1}\}$ where k is the smallest integer such that $(\sigma^{-1}\circ\alpha)^k(b)=b$ (resp. $(\sigma^{-1}\circ\alpha)^k(c)=c$) and,

$$\forall i\in\{0,\ldots,k-1\},\qquad b_i=(\sigma^{-1}\circ\alpha)^i(b)\quad\text{and}\quad c_i=(\alpha\circ\sigma)^i(c).$$

Since $\sigma^{-1}\circ\alpha(f)=f$, we have $\sigma(f)=\alpha(f)$.

(i) If ψ is the mapping of f into $\alpha(f)=\sigma(f)$ such that

$$\forall i\in\{0,\ldots,k-1\},\qquad \psi(b_i)=\sigma(c_i)\quad\text{and}\quad\psi(c_i)=\sigma(b_i),$$

the restrictions of ψ to $\mathrm{df}(b)$ and $\alpha(\mathrm{df}(c))$ and to $\mathrm{df}(c)$ and $\alpha(\mathrm{df}(b))$ are bijective. Moreover,

$$\alpha(\mathrm{df}(b))\cap\alpha(\mathrm{df}(c))\neq\emptyset\ \text{ if and only if }\ \mathrm{df}(b)\cap\mathrm{df}(c)\neq\emptyset$$

that is, if and only if $\mathrm{df}(b)=\mathrm{df}(c)$. Thus ψ is a bijection.

(ii) $\forall i\in\{0,\ldots,k-1\}$,

$$\sigma^{-1}\circ\psi(b_i)=c_i=\psi^{-1}\circ\sigma(b_i)$$

and

$$\psi^{-1}\circ\alpha(b_i)=\psi^{-1}\circ\sigma(b_{i+1})=c_{i+1}=\alpha\circ\sigma(c_i)=\alpha\circ\psi(b_i)$$

and similarly,

$$\sigma^{-1}\circ\psi(c_i)=\psi^{-1}\circ\sigma(c_i)\quad\text{and}\quad\psi^{-1}\circ\alpha(c_i)=\alpha\circ\psi(c_i).$$

(iii) By hypothesis $\psi(b)=\sigma(c)$ and $\psi(c)=\sigma(b)$ and, by (ii), if $\psi(b_i)=\sigma(c_i)$ and $\psi(c_i)=\sigma(b_i)$, then

$$\sigma^{-1}\circ\psi(b_{i+1})=\psi^{-1}\circ\sigma(b_{i+1})=\psi^{-1}\circ\alpha(b_i)=\alpha\circ\psi(b_i)=\alpha\circ\sigma(c_i)=c_{i+1}.$$

Thus $\psi(b_{i+1})=\sigma(c_{i+1})$ and similarly $\psi(c_{i+1})=\sigma(b_{i+1})$. The uniqueness of ψ follows. □

Definition 5.2. (i) If two darts b and c of a map $C=(B,\alpha,\sigma)$ verify $|\mathrm{df}(b)|=|\mathrm{df}(c)|$ the mapping ψ of Lemma 5.1 is said to be *associated* with (b,c) and if also,

$$\forall x\in f=\mathrm{df}(b)\cup\mathrm{df}(c),\quad \psi(x)\notin\{\alpha(x),\sigma(x)\}$$

darts b and c are said to be *equivalent*.

(ii) If the hypotheses of (i) are true, the transformation such that

$$B'=B\backslash\alpha(f),\qquad\forall x\in B'\backslash f,\ \alpha'(x)=\alpha(x)\quad\text{and}\quad\sigma'(x)=\sigma(x)$$

and

$$\forall x\in f\backslash\alpha(f),\ \alpha'(x)=\alpha\circ\psi(x)\quad\text{and}\quad\sigma'(x)=\sigma\circ\psi(x),$$

is called the *merging of map C according to (b,c)*.

Remark. Fontet [10] has defined a notion of resticking for two disjoint maps C_1 and C_2 according to a direct face f_1 of C_1 and a retrograd face f_2 of C_2 by identifying every dart of f_1 (resp. $\alpha_1(f_1)$) with a dart of f_2 (resp. $\alpha_2(f_2)$). The notion of merging realizes this resticking when applied to the disjoint union of C_1 and C_2 and generalizes it.

Example 5.3. If C is the map drawn by Fig. 10a and if $b=1$ and $c=1'$, then $\psi(1)=-2'$, $\psi(2)=-3'$, $\psi(3)=-4'$, $\psi(4)=-1'$, $\psi(1')=-4$, $\psi(2')=-1$, $\psi(3')=-2$ and $\psi(4')=-3$. The transformed map of C by merging according to $(1,1')$ is drawn by Fig. 10b.

Proposition 5.4. *If C' is obtained from a map C by merging according to a pair (b, c) of equivalent darts, then C' is a map and there exists a one-to-one mapping of the set of all direct faces of C distinct from* $\mathrm{df}(b)$ *and from* $\mathrm{df}(c)$ *onto the set of all direct faces of C'.*

Proof. (i) Since $\alpha(f)=\sigma(f)$, $\alpha\circ\sigma(f)=\alpha^2(f)=f$.

$$B_1=B'\backslash f\cap B' \quad \text{and} \quad B_2=f\cap B'=f\backslash f\cap\alpha(f)$$

constitute a partition of B'. $\psi(B_2)\subset\psi(f)=\alpha(f)$ is disjoint from B_1. Thus $\forall\beta\in\{\alpha,\sigma\}$, the mapping $\beta'=\beta|B'$ which extends the restrictions $\beta|B_1$ and $\beta\circ\psi|B_2$ is one-to-one. $\forall x\in B_1$, $\beta(x)\notin\beta(f)=\psi(f)$ and hence $\beta(x)\in B'$. $\forall x\in B_2$, $\beta\circ\psi(x)\in B'$ since $\psi(B_1\cap\psi^{-1}(f))=f\backslash f\cap\alpha(f)$ and $\psi(B_2)$ are disjoint and hence $\psi(x)\notin f$. Thus $\beta'(B')\subset B'$ and, since B' is finite, α' and σ' are permutations of B'.

(ii) $\forall x\in B_1$, $\alpha'(x)=\alpha(x)\in\alpha(B_1)\subset\alpha(B')=B\backslash f$ and thus $\alpha'^2(x)=\alpha^2(x)=x$.

$$\forall x\in B_2, \qquad \alpha(x)\in\alpha(B_2)\subset\alpha(f), \quad \alpha'(x)=\alpha\circ\psi(x)\in\alpha\circ\psi(f)=f$$

since $\alpha(f)=\psi(f)$ and

$$\alpha'^2(x)=\alpha\circ\psi\circ\alpha\circ\psi(x)=\alpha\circ\psi\circ\psi^{-1}\circ\alpha(x)=\alpha^2(x)=x$$

by Lemma 5.1. Moreover, $\forall x\in B_2$, $\alpha'(x)=\alpha\circ\psi(x)\neq x$ since $\psi(x)\neq\alpha(x)$. It follows that C' is a map.

(iii) The mapping δ from $B\backslash f$ into B' such that

$$\forall x\in(B\backslash f)\cap B', \quad \delta(x)=x$$

and

$$\forall x\in\alpha(f)\cap(B\backslash f)=\psi(B_2), \quad \delta(x)=\psi^{-1}(x)$$

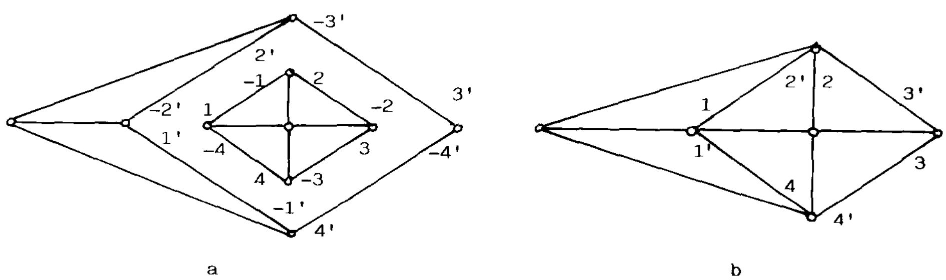


Fig. 10.

is bijective. Moreover, if $y=\sigma^{-1}\circ\alpha(x)$, then $\sigma'(\delta(y))=\sigma(y)=\alpha(x)=\alpha'(\delta(x))$ and thus $\delta(y)=\sigma'^{-1}\circ\alpha'(\delta(x))$. Since $\forall x\in B\backslash f$, $\mathrm{df}(x)\subset B\backslash f$, it follows that $\delta(\mathrm{df}(x))=\langle\sigma'^{-1}\circ\alpha'\rangle(\delta(x))$. The mapping $\mathrm{df}(x)\to\delta(\mathrm{df}(x))$ is then a one-to-one mapping of the set of direct faces of C distinct from $\mathrm{df}(b)$ and from $\mathrm{df}(c)$ onto the set of direct faces of C'. □

Definition 5.5. (i) $f=\mathrm{df}(b)\cup\mathrm{df}(c)$ is said to be *elementary* (resp. *simple*) if neither $\mathrm{df}(b)$ nor $\mathrm{df}(c)$ passes two times by the same vertex (resp. edge) of C, that is, $\forall x, y\in\mathrm{df}(b)$ (resp. $\mathrm{df}(c)$) $\langle\sigma\rangle(x)=\langle\sigma\rangle(y)$ (resp. $\langle\alpha\rangle(x)=\langle\alpha\rangle(y)$) implies $x=y$. If f is elementary (resp. simple) then,

$$\forall x\in\mathrm{df}(b),\ \ \mathrm{df}(b)\cap\langle\sigma\rangle(x)=\{x\}\quad(\text{resp. } a(b)\cap\langle\sigma\rangle(x)=\{x\})$$

and

$$\forall x\in\mathrm{df}(c),\ \ \mathrm{df}(c)\cap\langle\sigma\rangle(x)=\{x\}\ \ (\text{resp. } a(c)\cap\langle\sigma\rangle(x)=\{x\}).$$

(ii) If $x\in f$, we say that f *splits* $\langle\sigma\rangle(x)$ if $\langle\sigma\rangle(x)\backslash\alpha(f)$ is decomposed in at least two orbits for $\langle\sigma'\rangle$.

(iii) Two darts b and c are said to be *similar* if they are equivalent and if $\forall x\in f\cap\alpha(f)$, $\psi(x)=x$ and $\forall x\in f$ such that $\psi(x)\notin\langle\sigma\rangle(x)$, $\langle\sigma\rangle(x)\cap f=\{x\}$.

Lemma 5.6. *Let b and c be two similar darts such that $f=\mathrm{df}(b)\cup\mathrm{df}(c)$ is elementary.*

(i) *$\forall x\in f$ such that $\psi(x)\notin\langle\sigma\rangle(x)$,*

$$\begin{aligned}\langle\sigma'\rangle(x)&=\langle\sigma\rangle(x)\cup\langle\sigma\rangle(\psi(x))\backslash\alpha(f)\\&=\langle\sigma\rangle(x)\cup\langle\sigma\rangle(\psi(x))\backslash\{\sigma(x),\psi(x)\}.\end{aligned}$$

(ii) *$\forall x\in f$, f splits $\langle\sigma\rangle(x)$ if and only if $\psi(x)\in\langle\sigma\rangle(x)\backslash\{x,\sigma(x),\sigma^2(x)\}$ and then*

$$\begin{aligned}\langle\sigma\rangle(x)\backslash\alpha(f)&=\langle\sigma\rangle(x)\backslash\{\sigma(x),\psi(x)\}\\&=\langle\sigma'\rangle(x)\cup\langle\sigma'\rangle(\sigma^{-1}\circ\psi(x))\end{aligned}$$

with $\langle\sigma'\rangle(x)\neq\langle\sigma'\rangle(\sigma^{-1}\circ\psi(x))$.

Proof. (i) If $\psi(x)\notin\langle\sigma\rangle(x)$, then $y=\sigma^{-1}\circ\psi(x)\notin\langle\sigma\rangle(x)$ and, since $\psi(y)=\sigma(x)\in\langle\sigma\rangle(x)$, $\psi(y)\notin\langle\sigma\rangle(y)$ and, by Definition 5.5,

$$\langle\sigma\rangle(x)\cap f=\{x\}\quad\text{and}\quad\langle\sigma\rangle(y)\cap f=\{y\}.$$

Since

$$\sigma'(x)=\sigma\circ\psi(x)=\sigma^2(y)\quad\text{and}\quad\sigma'(y)=\sigma\circ\psi(y)=\sigma^2(x)$$

and, $\forall z\in\langle\sigma\rangle(x)\cup\langle\sigma\rangle(y)\backslash\{x,y\}$,

$$\begin{aligned}\sigma'(z)=\sigma(z),\qquad\langle\sigma'\rangle(x)&=\langle\sigma\rangle(x)\cup\langle\sigma\rangle(y)\backslash\{\sigma(x),\sigma(y)\}\\&=\langle\sigma\rangle(x)\cup\langle\sigma\rangle(\sigma^{-1}\circ\psi(x))\backslash\alpha(f).\end{aligned}$$

(see Fig. 11a).

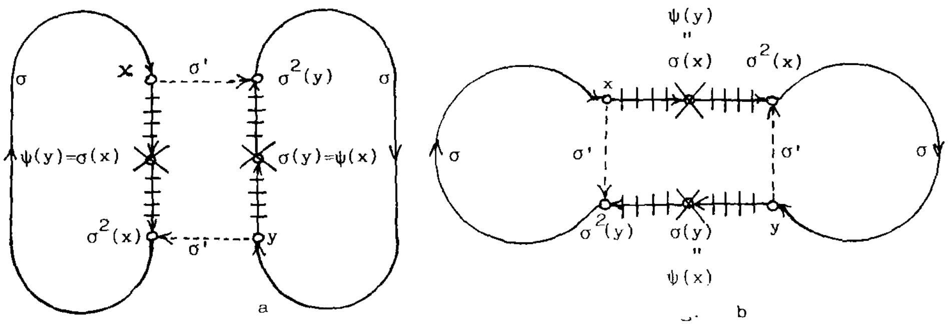


Fig. 11.

(ii) If $\psi(x) \in \langle\sigma\rangle(x)$, then

$$y = \sigma^{-1} \circ \psi(x) \in \langle\sigma\rangle(x) \quad \text{and} \quad \psi(y) = \sigma(x) \in \langle\sigma\rangle(x) = \langle\sigma\rangle(y).$$

Since f is elementary, $f \cap \langle\sigma\rangle(x) = \{x, y\}$. Thus, $\forall z \in \langle\sigma\rangle(x) \backslash \{x, y\}$, $\sigma'(z) = \sigma(z)$. Let $\langle\sigma\rangle(x) = \{x_0, \ldots, x_{t-1}\}$ where t is the smallest integer such that $\sigma^t(x) = x$ and $\forall i \in \{0, \ldots, t-1\}$, $x_i = \sigma^i(x)$.

(1) If $\psi(x) \in \langle\sigma\rangle(x) \backslash \{x, \sigma(x), \sigma^2(x)\}$, $\exists i \in \{3, \ldots, t-1\}$ such that $\psi(x) = x_i$ and then

$$y = \sigma^{-1} \circ \psi(x) = x_{i-1} \in \langle\sigma\rangle(x), \qquad \psi(y) = \sigma(x) = x_1 \in \langle\sigma\rangle(x),$$

$$\sigma'(x) = \sigma \circ \psi(x) = x_{i+1} \quad \text{and} \quad \sigma'(y) = \sigma \circ \psi(y) = x_2.$$

Since, $\forall z \in \langle\sigma\rangle(x) \backslash \{x, y\}$, $\sigma'(z) = \sigma(z)$,

$$\langle\sigma'\rangle(x) = \{x_0, x_{i+1}, \ldots, x_{t-1}\} \neq \langle\sigma'\rangle(y) = \{x_2, \ldots, x_{i-1}\}$$

and, since

$$\langle\sigma\rangle(x) \cap \alpha(f) = \langle\sigma\rangle(x) \cap \sigma(f) = \sigma(\langle\sigma\rangle(x) \cap f) = \sigma(\{x, y\})$$
$$= \{\sigma(x), \sigma(y)\} = \{\sigma(x), \psi(x)\},$$
$$\langle\sigma'\rangle(x) \cup \langle\sigma'\rangle(\sigma \circ \psi(x)) = \langle\sigma\rangle(x) \backslash \{\sigma(x), \psi(x)\}$$
$$= \langle\sigma\rangle(x) \backslash \alpha(f)$$

(see Fig. 11b).

(2) If $\psi(x) = x$, then $y = \sigma^{-1} \circ \psi(x) = x_{t-1}$ and, since $\sigma'(y) = x_2$, $\langle\sigma\rangle(x) \backslash \{x_0, x_1\} = \langle\sigma'\rangle(y)$ except if $\sigma^2(x) = x$.

The case of $\psi(x) = \sigma(x)$ is eliminated by hypothesis.

If $\psi(x) = \sigma^2(x)$, then $y = x_1$ and, since $\sigma'(x) = x_3$, $\langle\sigma\rangle(x) \backslash \{x_1, x_2\} = \langle\sigma'\rangle(x)$ except if $\sigma^2(x) = x$.

If $\psi(x) \notin \langle\sigma\rangle(x)$ then $\langle\sigma\rangle(x) \backslash \alpha(f) \subset \langle\sigma'\rangle(x)$ by (i).

Thus f does not split $\langle\sigma\rangle(x)$ in any of these cases. $\square$

Theorem 5.7. *If b and c are two similar darts of a map C and if the direct faces* $\mathrm{df}(b)$ *and* $\mathrm{df}(c)$ *are elementary and simple, then the genus of the transformed map C' of C by merging according to (b, c) is $g(C') = g(C) + e + g - h - k$ where, if $\alpha(f) = f$ then $e = 1$, otherwise $e = 0$ if $\langle\alpha, \sigma\rangle(b) = \langle\alpha, \sigma\rangle(c)$ and $e = -1$ in the opposite case, g is the number of connected components of $\langle\alpha, \sigma\rangle(b, c)\backslash\alpha(f)$ in C', h is the number of connected components of $f \cap \alpha(f)$ in C and k is the number of vertices of C which are split by f.*

Proof. (i) If $\alpha(f) \neq f$, then $\forall x \in f \cap \alpha(f)$, $\psi(x) = x$ and thus $\psi(\alpha(x)) = \alpha \circ \psi(x) = \alpha(x)$ by Lemma 5.1 and $\langle\alpha\rangle(x) \subset f \cap \alpha(f)$. Hence edge $\langle\alpha\rangle(x)$ is eliminated. Moreover, $\forall x \in f\backslash\alpha(f)$, $\psi(x) \notin \{x, \alpha(x)\}$ and hence $\langle\alpha\rangle(x) \neq \langle\alpha\rangle(\psi(x))$ whereas

$$\langle\alpha'\rangle(x) = \{x, \alpha \circ \psi(x)\} = \langle\alpha\rangle(x) \cup \langle\alpha\rangle(\psi(x))\backslash\{\alpha(x), \psi(x)\}.$$

Thus one edge is also eliminated in this case. It follows that, if m and m' are respectively the numbers of edges of C and of C' and if $t = |\mathrm{df}(b)|$, then $m - m' = t$.

By Lemma 5.6, $\forall x \in \mathrm{df}(b)$, the vertices $\langle\sigma\rangle(x)$ and $\langle\sigma\rangle(\psi(x))$ are merged in a single vertex except in the following cases:

Case (1): $\psi(x) = x$ *and* $\sigma^2(x) = x$. Then $\psi(\sigma(x)) = \psi(\sigma^{-1}(x)) = \sigma^{-1} \circ \psi(x) = \sigma^{-1}(x) = \sigma(x)$ and vertex $\langle\sigma\rangle(x) = \{x, \sigma(x)\}$ is also eliminated.

Case (2): $\psi(x) = x$ *and* $\sigma^2(x) \neq x$. In this case, vertex $\langle\sigma\rangle(x)$ is replaced by vertex $\langle\sigma'\rangle(\sigma^2(x)) = \langle\sigma\rangle(x)\backslash\{x, \sigma(x)\}$ and $\langle\sigma\rangle(x)$ is an initial endpoint of a path lying in $f \cap \alpha(f)$.

Case (3): $\psi(x) = \sigma^2(x)$ *and* $\sigma^2(x) \neq x$. In this case, vertex $\langle\sigma\rangle(x)$ is replaced by $\langle\sigma'\rangle(x) = \langle\sigma\rangle(x)\backslash\{\sigma(x), \sigma^2(x)\}$ and $\psi^{-1} \circ \sigma(x) = \sigma^{-1} \circ \psi(x) = \sigma(x)$. Thus $\psi(\sigma(x)) = \sigma(x)$ and hence $\langle\sigma\rangle(x)$ is a final endpoint of a path lying in $f \cap \alpha(f)$.

Case (4): $\psi(x) \in \langle\sigma\rangle(x)\backslash\{x, \sigma(x), \sigma^2(x)\}$. Then vertex $\langle\sigma\rangle(x)$ is split in two vertices of C'.

The respective numbers h_2 and h_3 of darts verifying the properties of Cases (2) and (3) are equal to h since every connected component of $f \cap \alpha(f)$ in C is a simple path. It follows that, if n and n' are respectively the numbers of vertices of C and C', then $n - n' = t - h_2 - h_3 - 2k = t - 2h - 2k$.

If $\langle\alpha, \sigma\rangle(b, c)\backslash\alpha(f)$ has g connected components in C', then $g - 1 + e$ is the difference between the number of connected components of C' and that of C. This proves that

$$\begin{aligned} g(C') &= g(C) + g - 1 + e + \tfrac{1}{2}[2 + m' - m + n - n'] \\ &= g(C) + e + g - h - k. \end{aligned}$$

(ii) If $\alpha(f) = f$ and if $t = |\mathrm{df}(b)|$ then, $\forall x \in f$, $\psi(x) = x$ and hence t edges and t vertices are eliminated. Hence $g(C') = g(C) + g$, which proves the general equality for $e = h = 1$ and $k = 0$.

Examples 5.8. (1) If C_1 is the map of Fig. 12a, $g(C_1) = 0$ and the darts 1 and 3′ are similar and ψ is such that $\forall x \in \{1, 2, 3\}$, $\psi(x) = -x'$ and $\psi(x') = -x$. Moreover, $f = \{1,$

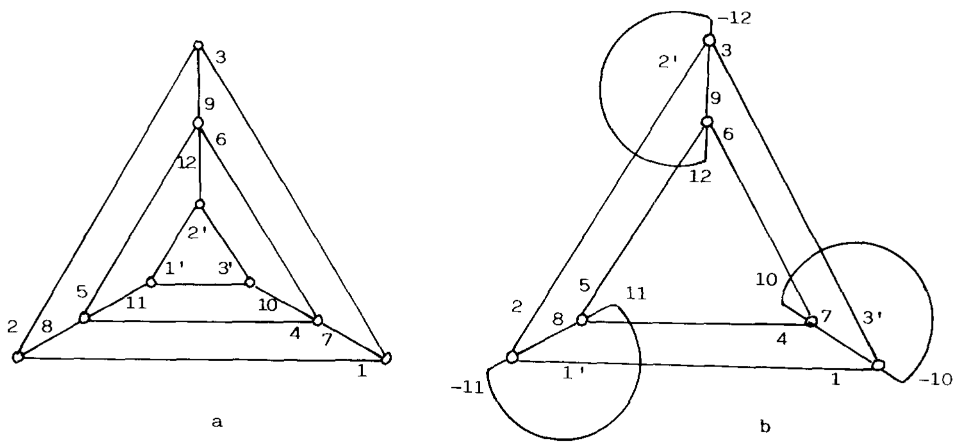


Fig. 12.

$2, 3, 1', 2', 3'\}$ does not split any vertex of C and verifies $f \cap \alpha(f) = \emptyset$. The transformed map C_1' of C_1 by merging according to $(1, 3')$ is drawn in Fig. 12b and $g(C_1') = 1$.

(2) If C_2 is the map of Fig. 13a, 1 and 4′ are similar darts and ψ verifies $\forall x \in \{1, 2, 3, 4\}$, $\psi(x) = -x'$ and $\psi(x') = -x$. Moreover $f = \{1, 2, 3, 4, 1', 2' = -2, 3' = -3, 4'\}$ is such that $f \cap \alpha(f) = \{2, -2, 3, -3\}$ but f does not split any vertex of C_2. Edges $\langle\alpha\rangle(2)$ and $\langle\alpha\rangle(3)$ and vertex $\langle\sigma\rangle(3)$ are eliminated and the transformed map C_2' of C_2 by merging according to $(1, 4')$ is also planar (see Fig. 13b).

(3) If C_3 is the planar map of Fig. 14a, 1 and 3′ are similar darts and ψ is such that $\forall x \in \{1, 2, 3\}$, $\psi(x) = -x'$ and $\psi(x') = -x$. $f = \{1, 2, 3, 1', 2', 3'\}$ splits vertex $\langle\sigma\rangle(1) = \{1, -3, 9, 3', -1', 8\}$ since

$$\psi(1) \in \langle\sigma\rangle(1) \backslash \{1, -3, 9\} \quad \text{and} \quad f \cap \alpha(f) = \emptyset.$$

The transformed map C_3' of C_3 by merging according to $(1, 3')$ is also planar (see Fig. 14b).

(4) If C_4 is the planar map of Fig. 15a, 1 and 4′ are similar darts and ψ is such that $\forall x \in \{1, 2, 3, 4\}$, $\psi(x) = -x'$ and $\psi(x') = -x$. $f = \{1, 2, 3, 4, 1', 2', 3', 4'\}$ splits

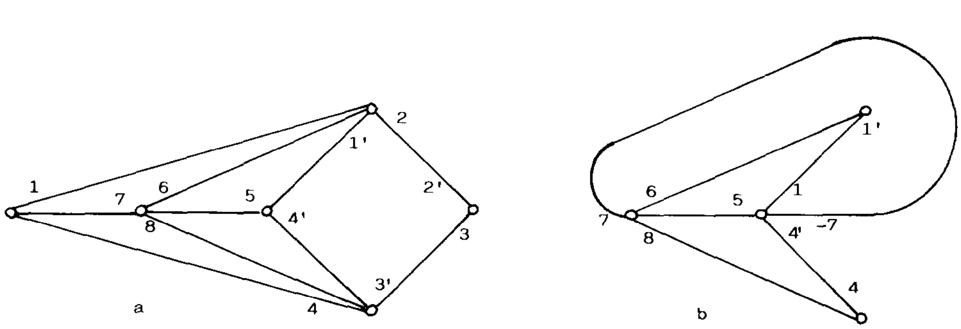


Fig. 13.

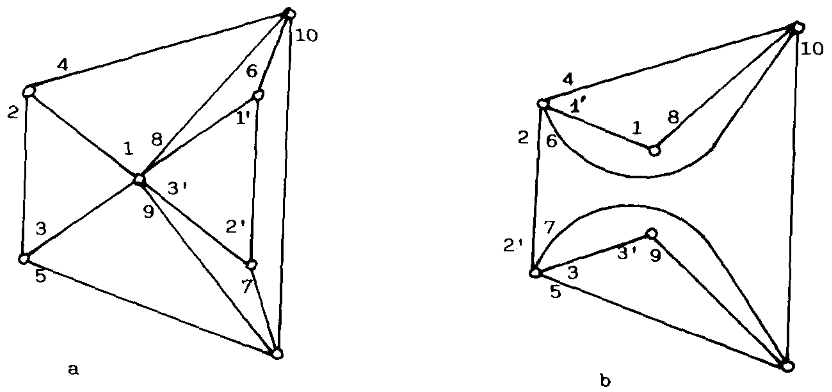


Fig. 14.

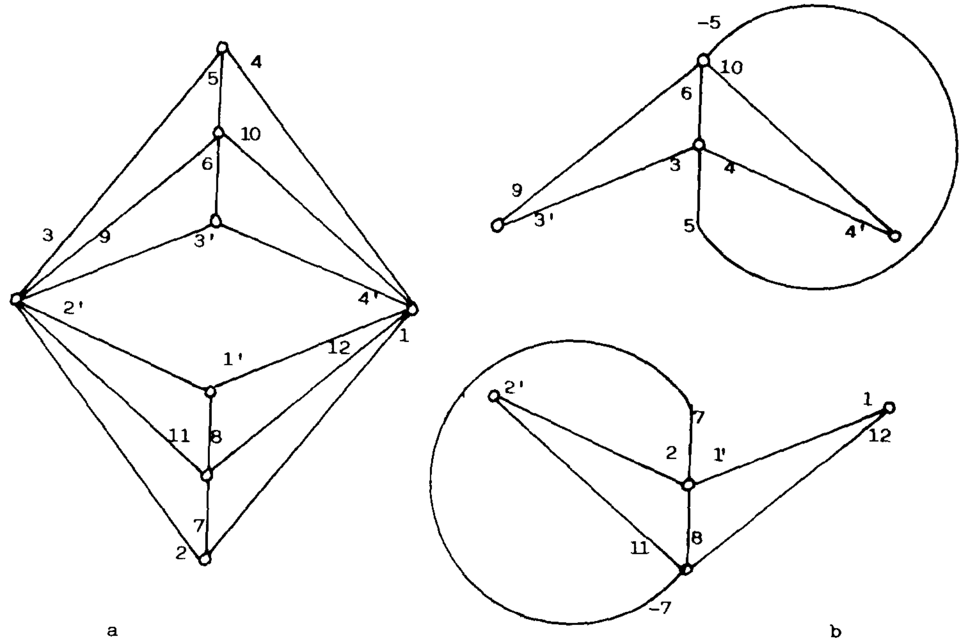


Fig. 15.

vertex $\langle\sigma\rangle(1)=\{1, -4, -10, 4', -1', 12\}$ since

$$\psi(1)=-1'\in\langle\sigma\rangle(1)\backslash\{1, -4, -10\}$$

and vertex $\langle\sigma\rangle(3)=\{3, -2, -11, 2', -3', 9\}$ since

$$\psi(3)\in\langle\sigma\rangle(3)\backslash\{3, -2, -11\}$$

giving respectively vertices $\langle\sigma'\rangle(1)=\{1, 12\}$ and $\langle\sigma'\rangle(4')=\{4', -10\}$ and vertices $\langle\sigma'\rangle(3')=\{3', 9\}$ and $\langle\sigma'\rangle(2')=\{2', -11\}$. The transformed map C_4' of C_4 by merging according to $(1, 4')$ is planar and has two connected components (see Fig. 15b).

6. Merging in pavings

Definition 6.1. Let $P=(B, \alpha, \sigma, \varphi)$ be a paving. If f is a face common to two pieces p_1 and p_2 of P, the transformation of P in $P'=(B', \alpha', \sigma', \varphi')$ such that $B'=B\backslash\varphi(f)$ and

$$\forall x\in B'\backslash f, \qquad \alpha'(x)=\alpha(x), \quad \sigma'(x)=\sigma(x) \quad \text{and} \quad \varphi'(x)=\varphi(x)$$

and

$$\forall x\in f\backslash\varphi(f), \qquad \alpha'(x)=\alpha\circ\varphi(x), \quad \sigma'(x)=\sigma\circ\varphi(x) \quad \text{and}$$

$$\varphi'(x)=\varphi^2(x)$$

is called the *merging of pieces* p_1 *and* p_2 *according to face* f or more simply the *merging of P along f.*

Remark. This extends to paving the merging in maps of Definition 5.2.

This notion of merging has been introduced by many authors [2, 13, 6] but the case where the two half-faces of f belong to the same piece is not dealt with. Nevertheless this omitted case is very natural and occurs necessarily if we merge successively along all the faces around an edge.

Example 6.2. (1) If P is the paving of Example 3.1, the pieces $p_1=p(1)$ and $p_2=p(7')$ admit $f=\{4, 5, 6, -4', -5', -6'\}$ as a common face. After merging of p_1 and p_2 according to f, we have $\alpha'=\alpha_1'\circ\alpha_2'\circ\alpha_3$ with $\alpha_1'=(4,-4')(5,-5')(6,-6')$, $\alpha_2'=\prod\{(x, -x);\ x\in\{1, 2, 3, 7', 8', 9'\}\}$ and the same α_3 as in Example 3.1,

$$\sigma'=(1,2,3)(-1,4,-7',-6')(-2,6,-9',-5')(-3,5,-8',-4')$$

$$\circ(7',8',9')\circ\sigma_3$$

with the same σ_3 as in Example 3.1. $P'=(B', \alpha', \sigma', \varphi')$ is then a paving which has 2 pieces $p'=p_1\cup p_2\backslash\varphi(f)$ and $p_3'=p(1'')$, 6 faces, 9 edges and 5 vertices. Thus $\chi(P')=0$ (see Fig. 16).

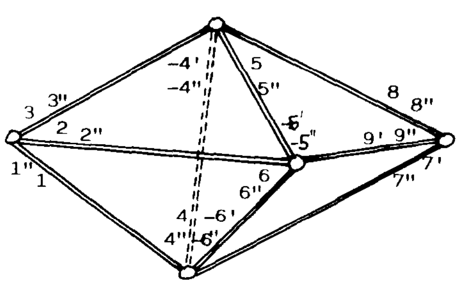


Fig. 16.

(2) If P is the paving of Example 3.4 (see Fig. 17a), $p_1 = p(1)$ and $p_2 = p(1')$ have $f = \{1, 2, 3, -1', -2', -3'\}$ as a common face. By merging of p_1 and p_2 according to f we have

- $\alpha' = \alpha'_1 \circ \alpha'_2$ with $\alpha'_1 = (1, -1')(2, -2')(3, -3')$ and $\alpha'_2 = \prod \{(x, -x);\ x \in \{4, 5, 4', 5'\}\}$,
- $\sigma' = (1, -5', -3', -5)(2, -1')(3, 4', -2', 4)(-4, 5)(-4', 5')$ and
- $\varphi' = (1)(2)(3)(-1')(-2')(-3')(4, 4')(-4, -4')(5, 5')(-5, -5')$.

$P' = (B', \alpha', \sigma', \varphi')$ is then a paving which has the only piece $p' = p_1 \cup p_2 \backslash \varphi(f)$, 2 *faces*

$$f'(1) = \mathrm{df}'(1) \cup \mathrm{df}'(-5) = \{1, 2, 4', 5', -4, -2', -1', -5\}$$

and

$$f'(3) = \mathrm{df}'(3) \cup \mathrm{df}'(4) = \{3, 4, 5, -3', -4', -5'\},$$

5 edges and 4 vertices. Thus $\chi(P') = 0$ (see Fig. 17b).

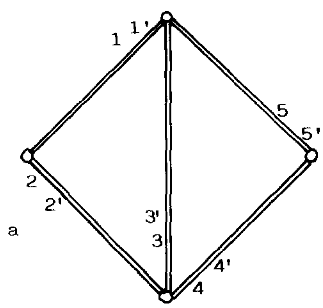
By merging of P' along $f' = f'(1)$ we have

$$B'' = B' \backslash \varphi'(f') = \{3, -3', -4, 4', -5, 5'\},$$

$\alpha'' = (3, -3')(-4, 4')(-5, 5')$, $\sigma'' = (3, 4')(5', -4)(-5, -3')$ and φ'' is the identity mapping of B''. Edges $a'(1) = \{1, -1'\}$ and $a'(2) = \{2, -2'\}$ and the vertex $s'(1) = \{-1', 2\}$ of P' are deleted. $P'' = (B'', \alpha'', \sigma'', \varphi'')$ is then a paving with a single piece, a single face, 3 edges and 3 vertices. Thus $\chi(P'') = 0$.

Proposition 6.3. *Let $P = (B, \alpha, \sigma, \varphi)$ be a paving. $P' = (B', \alpha', \sigma', \varphi')$ is obtained from P by merging two pieces p_1 and p_2 according to a common face f, then P' is a paving and the pieces of P' which are not included in $p' = p_1 \cup p_2 \backslash \varphi(f)$ are the pieces of P other than p_1 and p_2.*

Proof. (i) If $b \in p_1 \cap f$ and if $c = \varphi^{-1} \circ \sigma(b) = \sigma^{-1} \circ \varphi(b)$, then $c \in p_2 \cap f$ and $f = \mathrm{df}(b) \cup \mathrm{df}(c)$. By Lemma 1.4, $\varphi(f) = \alpha(f) = \sigma(f)$. The restriction ψ of φ to f and $\varphi(f)$ satisfies the hypotheses of Lemma 5.1 and, by Proposition 5.4, $C' = (B', \alpha', \sigma')$ is the map obtained from $C = (B, \alpha, \sigma)$ by merging according to a pair (b, c) of equivalent darts. Moreover, the proof that φ' is a permutation of B' is similar to that of Proposition 5.4 for α' and σ'.



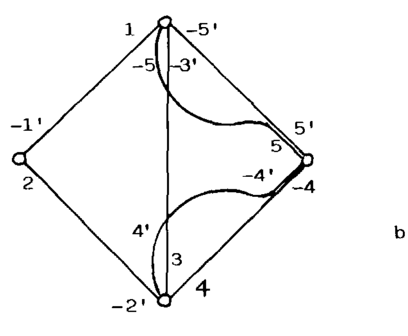


Fig. 17.

(ii) Let $B_1 = B' \backslash f \cap B'$ and $B_2 = f \cap B' = f \backslash f \cap \varphi(f)$.

$$\forall b \in B_1 \cap \varphi^{-1}(B_1), \qquad \varphi'(b) = \varphi(b) \in B_1 \quad \text{and} \quad \alpha' \circ \varphi'(b) = \alpha \circ \varphi(b).$$

Since $b \notin f$ implies that $\alpha \circ \varphi(b) \notin \alpha \circ \varphi(f) = f$, $\alpha \circ \varphi(b) \in B_1$ and, hence,

$$\varphi' \circ \alpha' \circ \varphi'(b) = \varphi \circ \alpha \circ \varphi(b) = \alpha(b) = \alpha'(b).$$

$$\forall b \in B_1 \cap \varphi^{-1}(f), \qquad \varphi(b) \in f \quad \text{and} \quad \alpha' \circ \varphi'(b) = \alpha \circ \varphi^2(b) \in \alpha \circ \varphi(f) = f.$$

Hence

$$\varphi' \circ \alpha' \circ \varphi'(b) = \varphi^2 \circ \alpha \circ \varphi^2(b) = \varphi \circ \alpha \circ \varphi(b) = \alpha(b) = \alpha'(b).$$

$\forall b \in B_2 = f \backslash f \cap \varphi(f)$, $\varphi'(b) = \varphi^2(b)$ and if $\varphi^2(b) \notin f$, then $\alpha' \circ \varphi'(b) = \alpha \circ \varphi^2(b) \notin f$ and

$$\varphi' \circ \alpha' \circ \varphi'(b) = \varphi \circ \alpha \circ \varphi^2(b) = \alpha \circ \varphi(b) = \alpha'(b)$$

otherwise

$$\alpha' \circ \varphi'(b) = \alpha \circ \varphi^3(b) \in \alpha \circ \varphi^3(f) = f$$

and

$$\varphi' \circ \alpha' \circ \varphi'(b) = \varphi^2 \circ \alpha \circ \varphi^3(b) = \alpha \circ \varphi(b) = \alpha'(b).$$

It follows that $\varphi' \circ \alpha' \circ \varphi' = \alpha'$.

(iii) $\forall b \in B_1$, $\varphi'(b) = \varphi(b) \in B'$ and hence $\varphi(b) \notin \varphi(f)$ and $\sigma^{-1} \circ \varphi(b) \notin \varphi^{-1} \circ \varphi(f) = f$. Thus

$$\sigma'^{-1} \circ \varphi'(b) = \sigma^{-1} \circ \varphi(b) \quad \text{and} \quad \varphi' \circ \sigma'^{-1} \circ \varphi'(b) = \varphi \circ \sigma^{-1} \circ \varphi(b) = \sigma(b)$$
$$= \sigma'(b).$$

$\forall b \in B_2$, $\varphi(b) \in \varphi(f)$ and $\varphi'(b) = \varphi^2(b) \in B'$. Thus

$$\varphi'(b) \notin \varphi(f) = \sigma(f) \quad \text{and} \quad \sigma^{-1} \circ \varphi^2(b) \notin f,$$

and hence

$$\varphi' \circ \sigma'^{-1} \circ \varphi'(b) = \varphi \circ \sigma^{-1} \circ \varphi^2(b) = \sigma \circ \varphi(b) = \sigma'(b).$$

It follows that $\varphi' \circ \sigma'^{-1} \circ \varphi' = \sigma'$.

(iv) Moreover,

$$\forall b \in B_1,\ \varphi'(b) = \varphi(b) \notin \{\alpha(b), \sigma(b)\} = \{\alpha'(b), \sigma'(b)\} \quad \text{and,}$$

$$\forall b \in B_2,\ \varphi'(b) = \varphi^2(b) \notin \{\alpha \circ \varphi(b), \sigma \circ \varphi(b)\} = \{\alpha'(b), \sigma'(b)\}.$$

Hence, P is a paving.

(v) Since $f \subset p_1 \cup p_2$, $\varphi(f) = \alpha(f) \subset p_1 \cup p_2$. For every piece p of P other than p_1 and p_2, $p \cap f = p \cap \varphi(f) = \emptyset$ and, hence, $p \subset B'$ and,

$$\forall x \in p, \qquad \alpha'(x) = \alpha(x) \quad \text{and} \quad \sigma'(x) = \sigma(x)$$

and hence p is also a piece of P'. This proves also that $p' = p_1 \cup p_2 \backslash \varphi(f)$ is stable for $\langle \alpha', \sigma' \rangle$, i.e. either p' is a piece or it is a union of pieces in P'. □

Definition 6.4. (i) A face $f(b)$ is said to be *elementary* (resp. *simple*) if the half-face $\mathrm{df}(b)=\langle\sigma^{-1}\circ\alpha\rangle(b)$ (or the half-face $\mathrm{df}(\varphi^{-1}\circ\sigma(b)))$ does not pass twice in some vertex (resp. edge) of P, i.e., $\forall x, y\in \mathrm{df}(b)$, $s(x)=s(y)$ implies $x=y$ (resp. $a(x)=a(y)$ implies $x=y$). If f is elementary,

$$\forall x\in f,\quad f\cap s(x)=\{x, \varphi^{-1}\circ\sigma(x)\}.$$

If f is simple,

$$\forall x\in f,\quad f\cap a(x)=\{x, \alpha\circ\varphi(x)\}.$$

(ii) Let x be a dart of a face f of P.

As in Definition 5.5, we say that f *splits* $\langle\sigma\rangle(x)$ if $\langle\sigma\rangle(x)\backslash\varphi(f)$ is divided into at least two orbits of $\langle\sigma'\rangle$.

We also say that f *splits the vertex* $s(x)$ if $s(x)\backslash\varphi(f)$ is divided into at least two vertices of P'.

Remark. Definition 6.4(i) is coherent with Definition 5.5(i) since if $f(b)$ is an elementary face of P, $\forall x, y\in\mathrm{df}(b)$ such that $\langle\sigma\rangle(x)=\langle\sigma\rangle(y)$, $\emptyset\neq\langle\sigma\rangle(x)\subset s(x)\cap s(y)$ and hence $s(x)=s(y)$ and thus $x=y$. Similarly, if $\mathrm{df}(b)$ is simple, $\langle\alpha\rangle(x)=\langle\alpha\rangle(y)$ implies $x=y$. Nevertheless, Definition 5.5(i) is more general.

Lemma 6.5. (i) *If f is a simple face of a paving P, then $\forall x\in f\cap\varphi(f)$, $\varphi(x)=x$.*

(ii) *If f is simple and if $\{x, \sigma(x)\}\subset f\cap\varphi(f)$, then $s(x)=\{x, \sigma(x)\}$.*

Proof. (i) If $b\in f$ and if $c=\varphi^{-1}\circ\sigma(b)$, the half-faces $\mathrm{df}(b)$ and $\mathrm{df}(c)$ are such that

$$\mathrm{df}(b)\cap\alpha(\mathrm{df}(b))=\mathrm{df}(c)\cap\alpha(\mathrm{df}(c))=\emptyset$$

Since α is an involution,

$$\mathrm{df}(b)\cap\alpha(\mathrm{df}(c))=\emptyset \text{ if and only if } \mathrm{df}(c)\cap\alpha(\mathrm{df}(b))=\emptyset$$

and since $\varphi(f)=\alpha(f)$,

$$f\cap\varphi(f)\neq\emptyset \text{ if and only if } \mathrm{df}(b)\cap\alpha(\mathrm{df}(c))\neq\emptyset.$$

$\forall x\in\mathrm{df}(b)\cap\alpha(\mathrm{df}(c))$, $\{\alpha(x), \alpha\circ\varphi(x)\}\subset\mathrm{df}(c)$ since $\varphi(\mathrm{df}(b))=\alpha(\mathrm{df}(c))$. But $\{\alpha(x), \alpha\circ\varphi(x)\}\subset a(x)$ and f is simple, hence $\alpha\circ\varphi(x)=\alpha(x)$ and, since α is one-to-one $\varphi(x)=x$. Moreover, $\varphi(\alpha(x))=\varphi\circ\alpha\circ\varphi(x)=\alpha(x)$ and thus $\alpha(x)\in f\cap\varphi(f)$. This proves that every dart of $f\cap\varphi(f)$ is fixed by φ.

(iii) If $\{x, \sigma(x)\}\subset f\cap\varphi(f)$, by (i), $\varphi(x)=x$ and $\varphi(\sigma(x))=\sigma(x)$. Thus

$$\sigma^{-1}(x)=\sigma^{-1}\circ\varphi(x)=\varphi^{-1}\circ\sigma(x)=\sigma(x) \quad\text{and}\quad s(x)=\{x, \sigma(x)\}. \quad\square$$

Lemma 6.6. *Let x be a dart of an elementary and simple face f of a paving P.*

(i) *f splits $\langle\sigma\rangle(x)$ if and only if $\varphi(x)\in\langle\sigma\rangle(x)\backslash\{x, \sigma(x), \sigma^2(x)\}$ and then*

$$\langle\sigma\rangle(x)\backslash\varphi(f)=\langle\sigma'\rangle(x)\cup\langle\sigma'\rangle(\varphi^{-1}\circ\sigma(x)) \quad \textit{with}$$

$$\langle\sigma'\rangle(x)\neq\langle\sigma'\rangle(\varphi^{-1}\circ\sigma(x)).$$

(ii) *If f splits $s(x)$ then f also splits $\langle\sigma\rangle(x)$.*

(iii) *If f splits $s(x)$ then*

$$s'(x) \neq s'(\varphi^{-1} \circ \sigma(x)) \quad \textit{and} \quad s(x) \backslash \varphi(f) = s'(x) \cup s'(\varphi^{-1} \circ \sigma(x)).$$

Proof. Since f is elementary,

$$\forall x \in f, \qquad s(x) \cap f = \{x, \varphi^{-1} \circ \sigma(x)\} \quad \text{and}$$

$$s(x) \cap \varphi(f) = \varphi(s(x) \cap f) = \{\varphi(x), \sigma(x)\}.$$

Thus $\langle\sigma\rangle(x) \cap \varphi(f) \subseteq \{\varphi(x), \sigma(x)\}$. Let $y = \varphi^{-1} \circ \sigma(x)$.

(i) By Lemma 6.5, $\forall x \in f \cap \varphi(f)$, $\varphi(x) = x$. Moreover, $\forall x \in f$ such that $\varphi(x) \notin \langle\sigma\rangle(x)$, $\varphi^{-1} \circ \sigma(x) = \sigma^{-1} \circ \varphi(x) \notin \langle\sigma\rangle(x)$ and hence $\langle\sigma\rangle(x) \cap f = \{x\}$. If $b \in f$ and if $c = \varphi^{-1} \circ \sigma(b)$, the darts b and c are similar and f verifies Definition 5.5. Thus (i) follows from Lemma 5.6.

(ii) The darts $\varphi'(x) = \varphi^2(x)$, $\sigma'(x) = \sigma \circ \varphi(x)$, $\sigma'(y) = \sigma \circ \varphi(y) = \sigma^2(x)$ and $\varphi'(y) = \varphi^2(y) = \varphi \circ \sigma(x)$ lie in $s(x)$ and hence $s(x) \backslash \varphi(f)$ is stable for $\langle\sigma', \varphi'\rangle$.

$\forall z \in B'$, let $V(z) = \{\sigma(z), \sigma^{-1}(z), \varphi(z), \varphi^{-1}(z)\} \backslash \varphi(f)$.

(1) If $\varphi(x) \notin \langle\sigma\rangle(x)$, then $y \in \langle\sigma'\rangle(x) \subset s'(x)$ by Lemma 5.6 and hence

$$V(\varphi(x)) = \{\sigma \circ \varphi(x), y, \varphi^2(x), x\} = \{\sigma'(x), y, \varphi'(x), x\} \subset s'(x)$$

and

$$V(\sigma(x)) = \{\sigma^2(x), x, \varphi \circ \sigma(x), y\} = \{\sigma'(y), x, \varphi'(y), y\} \subset s'(x).$$

Thus $s'(x) = s(x) \backslash \varphi(f)$.

(2) If $\varphi(x) = x \neq \sigma^2(x)$, then $\langle\sigma\rangle(x) \backslash \{x, \sigma(x)\} = \langle\sigma'\rangle(y) \subset s'(y)$ by the proof of Lemma 5.6 and, since $V(x) = \{y\}$ and

$$V(\sigma(x)) = \{\sigma^2(x), \varphi \circ \sigma(x), y\} = \{\sigma'(y), \varphi'(y), y\} \subset s'(y),$$

$$s'(y) = s(x) \backslash \varphi(f).$$

(3) If $\varphi(x) = \sigma^2(x) \neq x$, then $\langle\sigma\rangle(x) \backslash \{\sigma(x), \varphi(x)\} = \langle\sigma'\rangle(x)$ by the same proof and, since

$$V(\varphi(x)) = \{\sigma'(x), \varphi'(x), x\} \subset s'(x) \quad \text{and} \quad V(\sigma(x)) = \{x\},$$

$s'(x) = s(x) \backslash \varphi(f)$.

(4) If $\varphi(x) = \sigma^2(x) = x$, then $\varphi(\sigma^{-1}(x)) = \sigma^{-1}(\varphi(x)) = \sigma^{-1}(x)$ with $\sigma^{-1}(x) = \sigma(x)$ and $\langle\sigma\rangle(x) = s(x) = \{x, \sigma(x)\}$ is deleted by merging.

Hence f does not split $s(x)$ in each of these cases and since $\varphi(x) \neq \sigma(x)$ it follows that if f splits $s(x)$ then f also splits $\langle\sigma\rangle(x)$.

(iii) If f splits $s(x)$, then $\varphi(x) \in \langle\sigma\rangle(x) \backslash \{x, \sigma(x), \sigma^2(x)\}$ by (i) and (ii) and $y \notin \langle\sigma'\rangle(x)$ by Lemma 5.6. Since

$$V(\varphi(x)) = \{\sigma'(x), y, \varphi'(x), x\} \subset s'(x) \cup s'(y) \quad \text{and}$$

$$V(\sigma(x)) = \{\sigma'(y), x, \varphi'(y), y\} \subset s'(x) \cup s'(y),$$

$$s'(x) \neq s'(y) \quad \text{and} \quad s(x) \backslash \varphi(f) = s'(x) \cup s'(y). \quad \square$$

Theorem 6.7. *Let P be a paving and f an elementary and simple face common to two pieces p_1 and p_2 of P and let P' be the transformed paving of P by merging along f. Then*

$$\chi(P') = \chi(P) + e + g - h - k'$$

where $e = -1$ if $p_1 \neq p_2$, $e = 0$ if $p_1 = p_2$ and $\varphi(f) \neq f$, and $e = 1$ if $p_1 = p_2$ and $\varphi(f) = f$; g is the number of connected components of the map $p' = p_1 \cup p_2 \backslash \varphi(f)$ in P'; h is the number of connected components of $f \cap \varphi(f)$ in P and k' is the number of vertices of P which are split by f.

Proof. (i) If $p_1 \neq p_2$, since f is simple, $f \cap \varphi(f) = \emptyset$ and $\forall x \in f$, $\varphi(x) \notin \langle\sigma\rangle(x)$ and hence $h = k' = 0$ and $e = -1$. Moreover, since

$$\langle\sigma'\rangle(x) = \langle\sigma\rangle(x) \cup \langle\sigma\rangle(\varphi(x)) \backslash \varphi(f)$$

by Lemma 5.6, p' is connected and $g = 1$. By Proposition 5.4, P' has one face less than P. It follows that $\chi(P') = \chi(P) = \chi(P) + e + g - h - k'$.

(ii) If $p_1 = p_2$, let $p = p_1 = p_2$.

(1) If $\varphi(f) \neq f$, the pieces of P other than p are also pieces of P' and p' is a disjoint union of g pieces of P'. As in (i), P' has one face less than P and, by Lemma 6.5, every connected component of $f \cap \varphi(f)$ is a path and deletes 1 from the characteristic since if t is the length of this path, then t edges and $t-1$ vertices are deleted.

For every vertex $s(x)$ split by f, $s(x) \backslash \varphi(f)$ is divided in two vertices of P' by Lemma 6.6; such a vertex deletes 1 from the characteristic. Since $e = 0$ in this case, it follows also that $\chi(P') = \chi(P) + e + g - h - k'$.

(2) If $\varphi(f) = f$ with $f \neq p$, $\forall x \in f$, $\varphi(x) = x$ by Lemma 6.5 and if $t = |\mathrm{df}(b)|$, then t edges and t vertices are deleted by merging of P along f. Thus $\chi(P') = \chi(P) + g$ with $e = h = 1$ and $k' = 0$.

(3) If $f = p$, $\varphi(f) = \alpha(f) = f$ and piece p is simultaneously deleted with f. Thus $\chi(P') = \chi(P)$ with $e = h = 1$ and $g = k' = 0$. $\square$

Examples 6.8. If P is the paving of Fig. 18a, $\sigma = \sigma_1 \circ \sigma_2$ with

$$\begin{aligned}\sigma_1 = {} & (1, 4, 5, 6, 18, 16, 2, 3)(-1, -12, 7)(-2, -10, 8, 11) \\ & \circ (-3, -11, 9, 12)(-4, -7, -15)(-5, 15, -9, -14) \\ & \circ (-6, 14, -8, -13)(-16, 17, 10)(-18, 13, -17)\end{aligned}$$

and

$$\begin{aligned}\sigma_2 = {} & (1', 3', 2')(-1', 10', 7', -12')(-2', 11', 8', -10') \\ & \circ (-3', 12', 9', -11')(4', 6', 5')(-4', -15', -7', 13') \\ & \circ (-5', -14', -9', 15')(-6', -13', -8', 14'),\end{aligned}$$

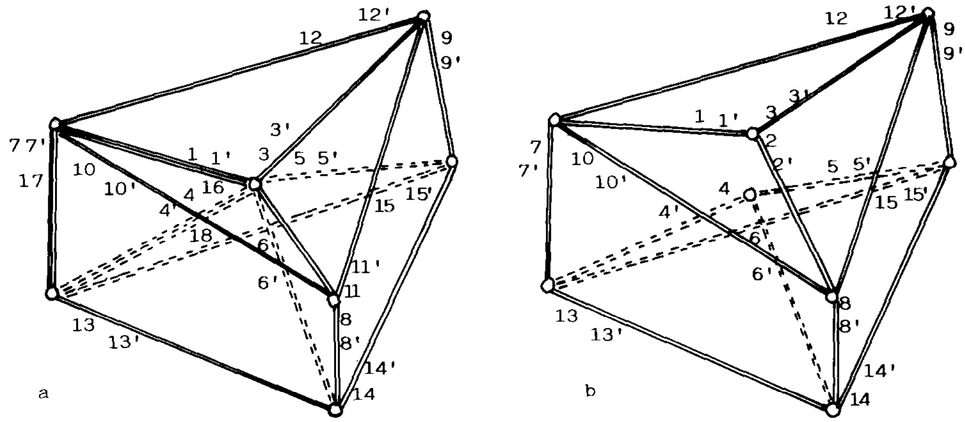


Fig. 18.

$\varphi = \varphi_1 \circ \varphi_2$ with

$$\varphi_1 = (1, 16, 1')(4, 4', 18)(7, 17, 7')(-1, -1', -16)$$
$$\circ (-4, -18, -4')(-7, -7', -17)$$

and

$$\varphi_2 = \pi\{(x, x');\ x \in X \cup -X\} \quad \text{where } X = \{2, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15\}.$$

$f = \{1, 7, -4, -16, 18, -17\}$ is an elementary and simple face which splits the vertex $s(1) = \{1, 4, 5, 6, 18, 16, 2, 3, 1', 2', 3', 4', 5', 6'\}$ in two vertices $s'(1) = \{1, 2, 3, 1', 2', 3'\}$ and $s'(5) = \{5, 6, 18, 4', 5', 6'\}$. Paving P' transformed by merging along f, is such that $\sigma' = \sigma_1' \circ \sigma_2$ where

$$\sigma_1' = (1, 2, 3)(5, 6, 18)(-12, 7, 10, -16)(-2, -10, 8, 11)(-3, -11, 9, 12)$$
$$\circ (-4, 13, -17, -15)(-5, 15, -9, -14)(-6, 14, -8, -13)$$

and $\varphi = \varphi_1' \circ \varphi_2$ where

$$\varphi_1' = (1, 1')(7, 7')(-4, -4')(-16, -1')(18, 4')(-17, -7')$$

(see Fig. 18b). $\chi(P) = 0$.

Definition 6.9. Let $x \in f$. If f splits $\langle\sigma\rangle(x)$ but does not split $s(x)$, the vertex $s(x)$ is said to be *bound on* f.

Theorem 6.10. *Let P be a paving, P' be the transformed paving by merging along an elementary and simple face f and C and C' the respective underlying maps of P and P'. If k'' is the number of bound vertices on f then*

$$\chi(P') - g(C') = \chi(P) - g(C) + k''$$

Proof. By Theorem 6.7, $\chi(P') = \chi(P) + e + g - h - k'$ where $e = 1$ if $\varphi(f) = f$, $e = 0$ if $p_1 = p_2$ and $\varphi(f) \neq f$ and $e = -1$ if $p_1 \neq p_2$, g is the number of connected components of the map $p' = p_1 \cup p_2 \setminus \varphi(f)$, h is the number of connected components of $f \cap \varphi(f)$ and k' is the number of vertices of P which are split by f.

If $b \in f$ and if $c = \varphi^{-1} \circ \sigma(b)$, b and c are similar darts by the proof of Lemma 6.6. By Theorem 5.7, $g(C') = g(C) + e + g - h - k$ with the same e, g and h as above and where k is the number of $\langle\sigma\rangle(x)$ which are split by f.

By Lemma 6.6, the number of bound vertices on f is $k'' = k - k'$. It follows that $\chi(P') - g(C') = \chi(P) - g(C) + k''$. $\square$

Example 6.11. Let $P = (B, \alpha, \sigma, \varphi)$ be the paving of Fig. 19: $\sigma = \sigma_1 \circ \sigma_2$ with

$$\sigma_1 = (1, -2, -4, -2'', 1'', 3)(2, -1, 5)$$

and

$$\sigma_2 = (4, -5, -3, -6)(6, -1'', 2'')(1', 3', -4', -2', -4'', 3'')(-1', 6', 2', 5') \circ (4', -3', -5')(4'', -6', -3''),$$

$\varphi = \varphi_1 \circ \varphi_2$ with

$$\varphi_1 = (1, 1'', 1')(-1'', -1, -1')(2, 2'', 2')(-2'', -2, -2')$$

and

$$\varphi_2 = (3, 3', 3'')(-3'', -3', -3)(4, 4', 4'')(-4'', -4', -4) \circ (5, 5')(-5, -5')(6, 6')(-6, -6')$$

and $\forall x \in B$, $\alpha(x) = -x$.

The transformed paving $P' = (B', \alpha', \sigma', \varphi')$ of P by merging along the face $f = \{1, 2, -1'', -2''\}$ which is elementary and simple is such that $\sigma' = \sigma_1' \circ \sigma_2$ with

$$\sigma_1' = (1, 3)(-2'', -4)(2, 6, -1'', 5),$$

$\varphi = \varphi_1' \circ \varphi_2$ with

$$\varphi_1' = (1, 1')(2, 2')(-1', -1'')(-2', -2'')$$

and $\alpha'(x) = -x$ except for $\alpha'(1) = -1''$, $\alpha'(2) = -2''$, $\alpha'(-1'') = 1$ and $\alpha'(-2'') = 2$.

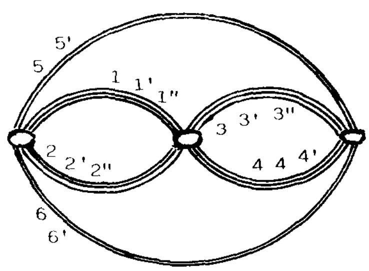


Fig. 19.

f splits $\langle\sigma\rangle(1)$ since $\varphi(1)=1''\in\langle\sigma\rangle(1)\backslash\{1,-2,-4\}$ but does not split

$$s(1)=\{1,-2,3,-4,1',-2',3',-4',1'',-2'',3'',-4''\}$$

since $s'(1)=s(1)\backslash\varphi(f)$. Thus vertex $s(1)$ is bound on f.

P admits 2 pieces, 6 faces, 6 edges and 3 vertices. Thus $\chi(P)=-1$ and $\chi(P')=0$ by Theorem 6.7. The underlying map of P has 2 connected components, 12 faces, 16 edges and 8 vertices. Thus $g(C)=0$ and $g(C')=0$ by Theorem 5.7. It follows that $\chi(P')-g(C')=\chi(P)-g(C)+1=0$ as stated by Theorem 6.10.

Conclusion

After generalities on a new notion of paving we dealt with merging in maps and merging in pavings and proved a relation between the characteristic of a paving and the genus of the underlying map. This result is fundamental for a subsequent paper on pavings [19]. Elbaz and the present author [8] have given an algorithm which constructs Voronoi diagrams in the plane by using maps and which is a variant of an algorithm of Guibas and Stolfi. We are generalizing it to the three-dimensional Euclidean space by using pavings and more precisely merging in pavings.

References

[1] S. Ansaldi, L. de Floriani and B. Falcidieno, Geometric modelling of solid objects by using graph representation, In: *Proc. SIGGRAPH'85 ACM*, San Francisco (1985) 131-139.

[2] D. Arquès and P. Koch, Définition et implémentation de pavages dans l'espace, *Theoret. Comput. Sci.* **77** (1990) 331-343.

[3] B.G. Baumgart, A polyhedron representation for computer vision, *AFIPS Nat. Comput. Conf.* **44** (1975) 589-596.

[4] E. Brisson, Representing geometric structures in d dimensions: topology and order, in: *Proc. 5th Ann. ACM Symp. on Computational Geometry* (1989) 212-227.

[5] R. Cori, Un code pour les graphes planaires et ses applications, *Asterisque* **27** (1975).

[6] D.P. Dobkin and M.J. Laszlo, Primitives for the manipulation of three-dimensional subdivisions, *Algorithmica* **4** (1989) 3-32.

[7] J.F. Dufourd, A topological map-based kernel for polyhedron modelers: Algebraic specification and logic prototyping, in: *Proc. Eurographics '89* (1989) 301-312.

[8] M. Elbaz and J.C. Spehner, Construction of Voronoi diagrams in the plane by using maps, *Theoret. Comput. Sci.* **77** (1990) 331-343.

[9] L. Euler, Elementa doctrinae solidorum, *Novi. Comm. Acad. Sci. Imp. Petropol.* **4** (1752-53) 109-140.

[10] M. Fontet, Connectivité des graphes et automorphismes des cartes: propriétés et algorithmes, Thèse Université de Paris 7, 1979.

[11] L. Guibas and J. Stolfi, Primitives for the manipulation of general subdivision and the computation of Voronoi diagrams, *ACM Trans. Graphics* **4** (2) (1985) 74-123.

[12] A. Jacques, Sur le genre d'une paire de substitutions, *Série A* **267** (1968) 625-627.

[13] P. Lienhardt, Extension of the notion of map and subdivision of the three-dimensional space, in: *Proc. STACS'88*, Lecture Notes in Computer Science, Vol. 294 (1988) 301-311.

[14] P. Lienhardt, Subdivision de surfaces, cartes et *S-V*-cartes. Public. No. 4, U.L.P. Strasbourg, 1988.

[15] P. Lienhardt, Subdivisions of n-dimensional spaces and n-dimensional generalized maps, in: *Proc. 5th Ann. ACM Symp. on Computational Geometry* (1989) 228-236.

[16] M. Mäntylä and R. Sulonen, GWB: A solid modeler with Euler operators, *CG&A* **2** (7) (1982).
[17] A. Requicha, Representation for rigid solids: theory, methods and systems, *Computing Surveys* **12** (4) (1980) 437–464.
[18] L. Schäfli, Theorie der vielfachen Kontinuität, *Denkschr. Schweiz Natur. Ges.* **38** (1901) 1–237.
[19] J.C. Spehner, On solid pavings, Public. Math-Inform. No. 49 University of Haute Alsace, 1988.
[20] W.T. Tutte, *Graph Theory* (Addison-Wesley, Reading, MA, 1984).